

Controlling an Inertial Wheel Pendulum : A Passivity Approach

A thesis submitted

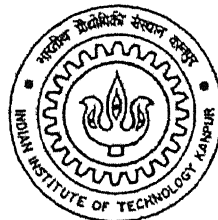
In partial fulfillment of the requirements

for the degree of

Master of Technology

by

Subhas Chandra Das



to the

DEPARTMENT OF ELECTRICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

July 2004.

TP
EE/2000/10
D26C

21 SEP 2004

गुरुचोत्तम कान्ही राम केलेकर पुस्तकालय
भारतीय प्रौद्योगिकी संस्थान मानपुर
यबापि क्र० A....148796....



A148796

9/7/04
k

CERTIFICATE

This is to certify that the thesis work entitled '**CONTROLLING AN INERTIAL WHEEL PENDULUM: A PASSIVITY APPROACH**' by *Subhas Chandra Das*, Roll No. *Y210439*, has been carried out under my supervision for the partial fulfilment of M.Tech. Degree in the department of Electrical Engineering, IIT, Kanpur and this work has not been submitted elsewhere for any other degree.

08 July 2004


Laxmidhar Behera
Dept. of Electrical Engineering,
Indian Institute of Technology, Kanpur

Acknowledgement

I would like to thank my guide Dr. Laxmidhar Behera. I really get admired to do work by his energetic and transparent personality. His way of thinking and understanding of physical systems in control point of view taught me how to deal with real time systems. He gave me enough freedom to experiment with my own ideas. He is a great source of inspiration; this work wouldn't have been possible without his constant support and motivation.

I would like to express my sincere gratitude to Dr. K. E. Hole for all the control systems theory that I learnt from him during the course work. Particularly "*Nonlinear system theory*" was of great help for my thesis.

I wish to convey my special thanks to my colleagues Prasad, Milind, Swagat, Raju and Vaibhav for their constant support through out my stay at IITK.

I am extremely grateful to my parents, whose inspiration and encouragement could bring me to the portals of my present achievements in academics. I am also grateful to Kunu with whose loving co-operation, I could succeed throughout.

At last my most special thanks goes to whoever concerned for the nice environment we have at IITK which is so conducive for pursuing good research.

Abstract

This thesis is concerned with nonlinear control of an Inertia Wheel Pendulum using passivity approach. The Inertia Wheel Pendulum (IWP) model is one of the simplest prototypes for underactuated systems to test variegated nonlinear control strategies.

IWP system dynamics has two equilibrium points of which the top unstable equilibrium point offers challenge to a control engineer. The control problem is often defined as swing-up the inertial wheel from downward bottom stable equilibrium position towards the top unstable equilibrium point where the wheel must stay stable forever. Implementation of a simple state feedback controller shows that the classical linearized controller can stabilize the pendulum at the top unstable equilibrium point if deflection angle $|x_{q1}| < 0.12 \text{radian}$ from the equilibrium position. Thus the problem is redefined if there exists a linearized control that can stabilize the pendulum when the deflection angle $|x_{q1}| < \frac{\pi}{2} \text{radian}$. It turns out that the control strategy using input-state feedback linearisation can stabilize the pendulum if the initial position of the pendulum $|x_{q1}| < \frac{\pi}{2} \text{radian}$. However, the original control problem is still not solved since the pendulum has to be brought from the downward bottom equilibrium position to the range defined by $|x_{q1}| < \frac{\pi}{2} \text{radian}$. Thus came the idea of swing up control as proposed by Spong [1].

A passivity based controller is designed where the IWP system is considered of consisting of two subsystems, namely, pendulum subsystem and wheel subsystem. The output variable is identified for each subsystem so that each subsystem remains passive from input to output. A control law is then proposed so that the combined system of pendulum and wheel is also passive. Further analysis shows that this proposed control law can only swing up the pendulum from the downward bottom stable equilibrium position to top unstable equilibrium position. A compensation term is added to take into account of friction during swing up which increases the amplitudes of swing up oscillations. However, the control law fails to stabilize the pendulum at the top unequilibrium position due to the fact that the controller brings the trajectory into a homoclinic orbit which intersects both stable and unstable equilibrium points. Thus the balancing controller is introduced to stabilize the linearized system at the top.

Another passivity based controller named Interconnection and Damping Assignment passivity based control is revisited. This method represents system dynamics in Hamiltonian

formulations instead of usual Euler Lagrangian (EL) formulation.. In Hamiltonian model, the system dynamics are parametrized by two matrices, called interconnection matrix and damping matrix. The controller is designed by choosing suitable closed loop interconnection and damping matrices for a system. The approach involves estimation of an extra energy function term H_a that must be added to open-loop energy function which makes the system passive in closed loop. The controller is parametrized in terms of partial differential terms involving estimated H_a . The stability of this controller algorithm has been proved using La Salle's invariance principle. Although this control algorithm can take the pendulum from its downward bottom equilibrium position to upward top equilibrium position at one go in simulation, the real time experiment can not be done using this algorithm. This is because the algorithm demands very high transient torque and speed demand for the inertial wheel which the available mechatronics control kit in the laboratory can not meet with. In essence, many possible theoretical algorithms can not be implemented in real time because of limitations in the hardware set up.

It could be ascertained during this work that dual mode controllers are best suitable for real-time implementation. This work ends with a novel proposal of swing up control using pumping of energy concept. Earlier, energy based swing up controllers have been proposed which could work in simulation, but failed during experimentation due to presence of friction. However, the proposed energy based swing up controller in this work works both in simulation and experimentation.

Contents

1	Introduction	1
1.1	Introduction	1
1.2	Feedback Linearisation	2
1.2.1	Input-Output Linearization	2
1.2.2	Input-State Linearization	3
1.3	Sliding Mode Control	3
1.4	Backstepping	5
1.5	Passivity Based Control	5
1.6	Underactuated systems	7
1.7	Present Work	8
2	Inertia Wheel Pendulum (IWP)	10
2.1	Introduction	10
2.2	The Model	10
2.3	The Experimental Setup	13
2.3.1	The Mechatronics Kit Hardware	13
2.3.2	The Software	14
2.3.3	System Parameters	14
2.3.4	Some Considerations during Experimentation	15
2.4	Linearisation	16
2.4.1	Analysis of Linearized Model	16
2.4.2	The State Feedback Control Law for Linearized System	17

2.4.3	Simulation and Experimental Results	19
2.5	Concluding Remarks	22
3	A hybrid Control of IWP	23
3.1	Introduction	23
3.2	Feedback Linearisation	23
3.2.1	Mathematical Preliminaries	24
3.2.2	feedback linearisation or Input-State Linearisation	27
3.2.3	Feedback linearisation of IWP	30
3.2.4	Control Design	32
3.2.5	Simulation and Experimental Results for Feedback Linearization . . .	33
3.3	Partial Feedback linearization	36
3.3.1	Analysis	36
3.3.2	Control Design	37
3.3.3	Simulation and Experimental Results of Complete Stabilization . . .	37
3.4	Concluding Remarks	40
4	Passivity Based Control of IWP	42
4.1	Introduction	42
4.2	Dissipativity and passivity	42
4.2.1	Passive systems and their properties	44
4.3	Passivity Based Control (PBC) of IWP	47
4.3.1	Effect of friction on Controller	50
4.4	Simulation and Experimental Results of Passivity Based Swing up	51
4.5	Experimental Results for Complete stabilisation	55
4.6	Concluding Remarks	55
5	IDA-PBC Control of IWP	57
5.1	Introduction	57
5.2	Port Controlled Hamiltonian systems	58
5.2.1	Port-controlled Hamiltonian model from EL and Hamiltonian model .	58

5.2.2	Basic properties of Port-controlled Hamiltonian systems	61
5.3	Interconnection and Damping Assignment PBC	62
5.4	The Control Problem in the Context of Passivity	64
5.5	IDA-PB Controller design	65
5.5.1	Application to mechanical systems	68
5.5.2	Inertia Wheel Pendulum System	73
5.5.3	Simulation Results for IDA-PBC on IWP	81
5.5.4	Simulation and Experimental results of Balancing IWP by IDA-PBC	84
5.5.5	Swing up Controller	88
5.5.6	Simulation and Experimental results for Complete stabilisation	91
5.5.7	Concluding Remarks	91
6	Conclusion	96
6.1	Summary	96
6.2	Scope of Future Work	97

List of Figures

2.1	The Inertia Wheel Pendulum	11
2.2	Simulation and Experimental Results For Linearised Control ($x_{q1} \sim t, \dot{x}_{q1} \sim t, x_{q2} \sim t$)	20
2.3	Simulation and Experimental Results For Linearised Control ($x_{q2} \sim t, u \sim t, \dot{x}_{q1} \sim x_{q1}$)	21
3.1	Simulation and Experimental Results of Feedback Linearization ($x_{q1} \sim t, \dot{x}_{q1} \sim t, x_{q2} \sim t$)	34
3.2	Simulation and Experimental Results of Feedback Linearization ($x_{q2} \sim t, u \sim t, \dot{x}_{q1} \sim x_{q1}$)	35
3.3	Simulation and Experimental Results of Complete Stabilisation ($x_{q1} \sim t, \dot{x}_{q1} \sim t, x_{q2} \sim t$)	38
3.4	Simulation and Experimental Results of Complete Stabilisation ($x_{q2} \sim t, u \sim t, \dot{x}_{q1} \sim x_{q1}$)	39
4.1	Output feedback passive system	45
4.2	Parallel interconnection systems	45
4.3	Feedback interconnection systems	46
4.4	IWP system as a parallel interconnection	47
4.5	Simulation and Experimental Results of Passivity based swing up ($x_{q1} \sim t, \dot{x}_{q1} \sim t, x_{q2} \sim t$)	52
4.6	Simulation and Experimental results of Passivity Based swing up ($\dot{x}_{q2} \sim t, u \sim t, E \sim t$)	53

4.7	Experimental results of Passivity Based swing up with LSF balancing control($x_{q1} \sim t, \dot{x}_{q1} \sim t, x_{q2} \sim t, \dot{x}_{q2} \sim t, u \sim t, E \sim t$)	56
5.1	Simulation results of IDA-PBC ($x_{q1} \sim t, \dot{x}_{q1} \sim t, x_{q2} \sim t, \dot{x}_{q2} \sim t, u \sim t, H_d \sim t$)	83
5.2	Simulation results of IDA-PBC ($H_p \sim t, \dot{x}_{q1} \sim x_{q1}$	84
5.3	Simulation and Experimental Results of Balancing IWP by IDA-PBC ($x_{q1} \sim t, \dot{x}_{q1} \sim t$)	86
5.4	Simulation and Experimental Results of balancing IWP by IDA-PBC ($x_{q2} \sim t, \dot{x}_{q2} \sim t, u \sim t$)	87
5.5	Simulation and Experimental Results of balancing IWP by IDA-PBC ($H_d \sim t, H_p \sim t, \dot{x}_{q1} \sim x_{q1}$)	88
5.6	Simulation and Experimental results for Complete Stabilization ($x_{q1} \sim t, \dot{x}_{q1} \sim t, x_{q2} \sim t$)	93
5.7	Simulation and Experimental results for Complete Stabilisation ($\dot{x}_{q2} \sim t, u \sim t, H_p \sim t$)	94
5.8	Simulation and Experimental results for Complete Stabilization ($H_d \sim t, \dot{x}_{q1} \sim x_{q1}$)	95

Chapter 1

Introduction

1.1 Introduction

Natural systems are inherently nonlinear. Studies in nonlinear systems have thus fascinated researchers from many fields. As one builds artificial systems mimicking natural systems, not only one faces the challenges of dealing with nonlinear dynamics, but also with the problem of dealing with numerous actuators (biological motor controls) that natural systems do have. Thus comes the motivation to design underactuated systems although such systems are even more difficult to control as compared to fully actuated nonlinear systems. One often comes across many underactuated systems examples such as underwater vehicles, helicopters, aircrafts, hovercrafts, satellites and walking robots. Although there exists variegated control algorithms for fully actuated nonlinear systems these algorithms are not directly applicable to underactuated systems. More over, a general framework has not evolved to describe all possible underactuated systems. Hence the control problem for an underactuated system that can be dealt with has to be system specific since it can not be generic at the present moment. One should be aware of various nonlinear control strategies already available in the literature before designing controllers for underactuated systems. Some of the nonlinear control techniques that are relevant to the design of controllers for underactuated systems are feedback linearization, sliding mode control, backstepping and passivity based controllers.

1.2 Feedback Linearisation

The idea of feedback linearization is to transform a nonlinear system dynamics into a linear one using (dynamic) feedback. The advantage is evident that once the system is transformed into linear one, all linear control techniques can be applied. The theory of linearization is developed from Differential Geometry (details in [2, 3]). There are basically two kinds of approaches to linearization, namely,

1. Input-Output linearization
2. Input-State linearization

1.2.1 Input-Output Linearization

Consider a single input single output (SISO) nonlinear system

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}\tag{1.1}$$

In the above equation (1.1) output is related to input through state. The objective is to bring out direct linear relationship between input and output,

$$y^{(m)} = v\tag{1.2}$$

where $v = \gamma(x, u)$, $m \in \mathbb{R}$ indicates differentiation by m times and $m \leq n$ and m is known as relative degree of the system (1.1).

If in the case $m < n$, then the transformed system (1.2) will have unobservable dynamics of order $n - m$. The unobservable dynamics, named as zero dynamics by Isidori et al. in [3], are unstable then this technique fails.

MIMO systems are also dealt in the same way with bit more complexity (see [4]). This technique has been successfully implemented in several practical applications.

1.2.2 Input-State Linearization

In this approach, the objective of linearization is achieved in two steps, State transformation and Input transformation. Moreover in this approach we are not concerned with output. The formal definition of the problem (in SISO case) is given in [2]. Consider the system (1.1a),

$$\dot{x} = f(x) + g(x)u$$

the objective is to find out a state and input transformations,

$$\begin{aligned} v &= q(x) + s(x)u \\ z &= T(x) \end{aligned} \tag{1.3}$$

so that the original system is transformed into linear one,

$$\dot{z} = Az + Bv \tag{1.4}$$

In fact in the equation (1.2), relative degree, $m = n$, order of the system (1.1) then Input-Output linearization leads to Input-State Linearization with state transformed from $x \rightarrow y$.

Control design using feedback linearization involves partial differential equations and solving these equations is really difficult particularly in the case of MIMO systems. Still the topic is under study and the research is underway to overcome the problem of solving partial differential equations.

1.3 Sliding Mode Control

The underlying principle behind Sliding Mode Control (SMC) is to alter the system dynamics via the application of a high speed switching control, hence it is also known as Variable Structure Control. In fact this approach is also a state feedback scheme but feedback gains are discontinuous functions of time.

This technique has drawn interest among control research community since the publication of the survey paper [5] by Vadim I. Utkin in 1977. This technique is pretty old but still

in application because of its advantages, simplicity and inherent ability to take care of uncertainties present in the model parameters once we know the bounds on uncertainty.

On the negative side it suffers from the so-called chattering phenomenon. Chattering phenomenon is generally perceived as motion which oscillates about the sliding manifold. Many analytical design methods were proposed to reduce the effects of chattering [6]. Secondly, the exact performance of zero error can not be achieved as it needs infinitely fast switching. So trade-off between the error tolerance and switching frequency is to be made. Here we give a brief idea about control design. Consider again the system (1.1).

$$\dot{x} = f(x) + g(x)u$$

The basic methodology is to choose a stable one dimensional manifold in the whole state space and choose the control input in such a way that the state evolves along the stable manifold called sliding surface. For the above system sliding surface can chosen to be

$$s = a_n \tilde{x}_n + a_{n-1} \tilde{x}_{n-1} + \cdots + a_1 \tilde{x}_1 = 0 \quad (1.5)$$

where tracking error, $\tilde{x}_j = x_j - x_j^d$ and x^d represents the desired trajectory. Note that the unique solution of first order differential equation is $\tilde{x} = 0$. Hence the problem is simplified as one has to select a suitable control input u so as to keep $s = 0$ to achieve the desired tracking performance. Let us select a control action such that

$$\frac{1}{2} \frac{d}{dt} s^2 \leq -\eta |s| \quad (1.6)$$

where η is strictly positive constant.

The above equation tells that if the system state is going away from the manifold $s = 0$, then the control action pushes the states towards the same manifold $s = 0$.

The sliding mode technique has been successfully applied to various non-linear systems like robot manipulators, control of automobiles, high-performance electric motors.

1.4 Backstepping

Backstepping is another alternative technique to design stabilizing controllers for nonlinear systems. This technique is motivated by the fact that first order systems are easy to deal with and can be used as starting points of recursive design for higher order systems.

A quantitative treatment of this technique is not provided here since the backstepping concept has been elaborated in chapter 2. For detailed treatment of backstepping technique one can refer the book by Krstic, Kanellakopoulos and Kokotovic [7]. The approach primarily makes use of Lyapunov stability theory and recursively designs controller starting from first order to higher order dynamics.

Backstepping finds great scope in the field of adaptive and robust control [8]. Neural network based robust backstepping techniques [9] have been proposed which eliminates the tedious calculation of regression matrices. In addition such neural control schemes do not demand that nonlinear systems should be linearly parametrizable - the dynamics is linear in terms of unknown parameters - for designing adaptive control schemes.

1.5 Passivity Based Control

Passivity is a fundamental property of many physical systems which may be roughly defined in terms of energy dissipation and transformation [10]. It is an inherent Input-Output property in the sense that it quantifies and qualifies the energy balance of a system when stimulated by external inputs to generate some output. Passivity is therefore related to the property of stability in an input-output sense, that is the system is stable if bounded input energy supplied to the system yields bounded output energy. Passivity based control is a methodology which consists in controlling a system with the aim at making the closed loop system passive.

The development of passivity and the connections with feedback stabilization of a system, from general operator theoretic point of view, were introduced in the work of Willems [11]. A different research line was initiated by work of Wu and Desoer [12]. The development of passivity concepts, in relation to stability, was undertaken in the work of Vidyasagar [13] and Zames [14]. The extension of Willem's results to the case of nonlinear systems, which

are affine in the control input, was given by Hill and Moylan in [15]. The idea of making a system passive by means of a state static feedback and all the related geometry was given in the work of Byrnes *et al* [16]. These results were later complemented, in connection with the concept of feedback positive systems, in the work of Kokotovic and Sussman [17]. Definite connections between passivity, Lyapunov-based control and inverse optimal feedback can be found out in the book by Sepulchre *et al* [18]. The conditions under which a nonlinear system is rendered passive, via state static state feedback were extended by Santosuosso [19] to the case of affine feedthrough system. The study of feedback dissipativity, for linear case, has been addressed in the work of Picci and Pinzoni (see [20]). The corresponding developments for the passivity of discrete time nonlinear systems were given by Lin and Lin and Byrnes in [21]. For a global perspective area these books can be referred, by Sepulchre *et al* [18], Ortega *et al* [22] and Khalil [23]. The passive system is a system that can not store more energy than the energy that is supplied to it. In other words passive system does not contain any energy source. Consider the generalised non-linear system

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + g(x(t))u(t) \\ y(t) &= h(x(t)) + j(x(t))u(t)\end{aligned}\tag{1.7}$$

where $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}^m$, and the nonlinear functions, defined by $f(\cdot), g(\cdot), h(\cdot), j(\cdot)$, are smooth. Let us call $S(u(t), y(t))$ the supply rate, such that $\forall u, \forall x(0)$

$$\int_0^t |S(u(t), y(t))| dt < \infty \quad t \in \mathbb{R}^+$$

Definition 1.1 (*Dissipativity*) A system of the form 1.7 is said to be dissipative with respect to the supply rate $S(u, y)$ if and only if there exists a storage function $H(x(t))$ defined as $H(x(t)) = K + P$, (where K is the kinetic energy and P is the potential energy of the system) such that

$$H(x(t)) \leq H(x(0)) + \int_0^t S(u(t), y(t)) dt\tag{1.8}$$

$\forall u$ and $\forall x(0)$.

This is called dissipation inequality, which means the storage energy function $H(x(t))$ at a future time t is not bigger than the sum of the available energy function $H(x(0))$ at an initial

time 0 plus the total energy $\int_0^t S(u(t), y(t))dt$ supplied to the system from external source in the interval $[0, t]$. There is no internal creation of energy.

Passive system represent an important subset of dissipative systems.

Definition 1.2 (*Passivity*) An EL system of the form is passive if it is dissipative with supply rate $S(u, y) = u^T y$. Because the supplied energy is greater than or equal to the total energy H , the passivity property can also be written as

$$H(x(T)) - H(x(0)) \leq \int_0^T u^T(t)y(t)dt \quad (1.9)$$

$\forall u(t), \forall T \geq 0$ and all $H(x(0))$

1.6 Underactuated systems

The formal definition of underactuated system is as follows [24] :

Definition 1.3 *Underactuated systems are the systems of the form*

$$\ddot{q} = f(q, \dot{q}) + G(q)u \quad (1.10)$$

where, q is the state vector of independent generalised coordinates, (also called as vector of configuration variables),

$f(\cdot)$ is the vector field representing the dynamics of the systems,

$\dot{q}$ is the generalised velocity vector,

G is the input matrix, and

u is a vector of generalised force

The dimension of q is defined as the degrees of freedom of the system in (1.10). In general (rather literally) a system is said to be underactuated if the external generalized forces are not able to command instantaneous accelerations in all directions in the configuration space, i.e. $rank(G) < dim(q)$. Feedback linearization, sliding mode control and backstepping

schemes as discussed earlier are not directly applicable to underactuated systems. The Lagrangian dynamics of these underactuated systems may contain feedforward nonlinearities [25], non-minimum phase zero dynamics, and other properties that place these class of systems in forefront of research in nonlinear control. Here arises the question that why to go for UAMS with all the complications involved. The reason is as follows:

1. In many practical cases actuators are expensive or heavy in weight, for these reasons some systems are underactuated by design.
2. In some cases due to failure of some actuator fully-actuated systems may also become underactuated and then appropriate control algorithm can be used to still meet the requirement.
3. Some physical properties like flexible (in joint) make the system underactuated inherently [26].
4. The working medium (water and space) adds one or more degrees of freedom where even fully actuated system behaves like an Underactuated System [27].

Thus there is a need to develop new control techniques applicable to underactuated nonlinear systems. The research is focused towards obtaining generalized control algorithms for underactuated nonlinear mechanical systems. Since generalized objective is difficult to achieve and also we are interested in stabilizing particular classes of mechanical systems, to simplify the task researchers have been studying simple mechanical systems. Some of the systems represent academic benchmarks and are part of a standard control laboratory like the inertia wheel pendulum, pendubot, furuta pendulum, the planar manipulator with springs between the links, the ball and beam and the PVTOL aircraft. In spite of they are simple mechanical plants, they represent a challenge to the non-linear control community.

1.7 Present Work

The Inertia Wheel Pendulum (IWP) is one of the simplest prototypes to study underactuated systems. Usually researchers have adopted dual mode controller to control an inertia

wheel Pendulum. Although there are attempts to control IWP models using single control algorithm[28, 29], such control algorithms are not implementable in real time due to severe transient demands on the wheel actuation torque and the wheel speed. The dual mode control consists of two steps - swing up and balancing control.

During swing up, the objective is to develop the oscillations that steadily increases in amplitude so as to reach very near to top unstable equilibrium point. In other words, energy is pumped into the system during swing up phase to accelerate the system towards high amplitude oscillations. The energy that is being pumped in should be properly shaped so as to result in controlled oscillations for the pendulum to reach very close to the top unstable equilibrium point.

Three different dual mode control strategies have been discussed, each consists of a swing up control and a balancing control. More emphasis is given to passivity based or energy based control. First swing up control is based on partial feedback linearisation. Other two are energy based. The energy based swing up is achieved by shaping pendulum system energy using two different representation of system dynamics. The first passivity based controller is designed using the system dynamics expressed in terms of Euler-Lagrange equations of motions. The other control algorithm is designed using Hamiltonian system model.

The objective of balancing controller is to balance the pendulum at top unstable equilibrium point against disturbances. The first balancing control is a simple pole placement control by linearizing the system about unstable equilibrium point. The second one is based on input-state feedback linearization. And the third one is based on passivity approach in which the system is modeled as a Port Controlled Hamiltonian system with Dissipation (PCHD), the stabilization is achieved by shaping both kinetic and potential energy making use of interconnection of Hamiltonian structures and injecting damping into the system by negative passive output feedback.

The controllers have been implemented in real-time on a Mechatronics control kit supplied by Quanser Inc. Experimental results are quite satisfactory.

Chapter 2

Inertia Wheel Pendulum (IWP)

2.1 Introduction

In this chapter, the *Inertia Wheel Pendulum (IWP)* model has been revisited which was first introduced by Spong[1]. It is one of the simplest non-linear underactuated systems with non-holonomic constraints¹. The Inertia Wheel Pendulum is a physical pendulum with a symmetric disk attached to the end which is free to spin about an axis parallel to the axis of rotation of pendulum. The disk is actuated by a DC-motor and the coupling torque generated by the angular acceleration of the disk is used to actively control the system. The Inertia Wheel Pendulum can be thought of as a simple pendulum in parallel with a torque controlled lumped inertia. Due to simplicity it can be used to illustrate a wide range of advanced nonlinear control problems.

2.2 The Model

The Inertia Wheel Pendulum is a two degrees of freedom robot manipulator, of which pendulum constitutes the first link while rotating wheel is the second. The schematic diagram is shown in Figure 1.1. Since first link is provided with no actuator, the system is underactuated with one actuator at the rotating disk. The nonlinearity in system equations is due to the application of torque due to gravitational field. The equations of motion can be derived using

¹Constraints which are not of the form $h(x_{q1}, x_{q2}, t) = 0$. They involve derivatives and double derivatives of the angular variables x_{q1} and x_{q2} and are not integrable.

the Euler-Lagrange (EL) formulation. The following system parameters are identified which are involved in the system dynamics.

m_p Mass of Pendulum.

m_r Mass of wheel.

m Combined mass of pendulum and wheel.

J_p Moment of inertia of the pendulum about its center of mass.

J_r Moment of inertia of wheel about its center of mass.

l_p Distance from pivot to the center of mass of pendulum.

l_r Distance from pivot to the center of mass of wheel.

l Distance from pivot to the center of mass of pendulum and wheel.

ψ_p Angle of pendulum with the reference.

ψ_r Angle of rotor with reference (as shown in figure2.1).

τ Motor torque input applied on disk (as shown in figure2.1).

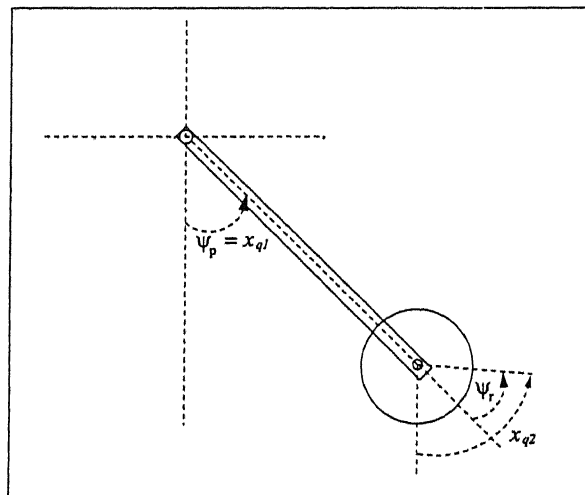


Figure 2.1: The Inertia Wheel Pendulum

The system parameters are redefined as follows:

$$\begin{aligned}
x_{q1} &= \psi_p \\
x_{q2} &= \psi_p + \psi_r \\
m &= m_p + m_r \\
ml &= m_p l_p + m_r l_r \\
J &= J_p + m_p l_p^2 + m_r l_r^2
\end{aligned} \tag{2.1}$$

The kinetic energy K of the system is the sum of pendulum kinetic energy and rotor kinetic energy and can be written as

$$K = \frac{1}{2} J \dot{x}_{q1}^2 + \frac{1}{2} J_r \dot{x}_{q2}^2 \tag{2.2}$$

The potential energy of the system P is

$$P = mgl(1 - \cos x_{q1}) \tag{2.3}$$

where the potential energy is zero when pendulum is hanging in the downward position. Finally the Lagrangian function L is given by

$$\begin{aligned}
L &= K - P \\
L &= \frac{1}{2} J \dot{x}_{q1}^2 + \frac{1}{2} J_r \dot{x}_{q2}^2 - mgl(1 - \cos x_{q1})
\end{aligned} \tag{2.4}$$

The equation of motions are given using EL formulation as follows:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\psi}} \right) - \frac{\partial L}{\partial \psi} = \tau \tag{2.5}$$

where $\psi = [\psi_p, \psi_r]^T$. Neglecting the frictional forces and electrical dynamics of the DC motor placed at the wheel, the torque vector is given as:

$$\tau = \begin{bmatrix} 0 \\ k_t I \end{bmatrix} \tag{2.6}$$

where k_t is the torque constant of the motor and I is the motor current. The first element of the torque vector is zero since the pendulum link is not actuated. Simplifying equation 2.5, and using transformations given in equation 2.1, the final form of equation of motion is given as follows:

$$\begin{aligned} J\ddot{x}_{q1} + mgl \sin x_{q1} &= -k_t I \\ J_r \ddot{x}_{q2} &= k_t I \end{aligned} \quad (2.7)$$

The equations (2.7) represent system dynamics when the reference for pendulum angle and rotor angle is along the vertical downward axis as shown in Figure 2.1. Instead if the reference is taken along vertical upward axis, then kinetic energy of the system remains same, while the potential energy becomes $P = mgl(\cos x_{q1} - 1)$, which is zero at vertical upward position of the pendulum. In this case, the EL equations of motions take the following form:

$$\begin{aligned} J\ddot{x}_{q1} - mgl \sin x_{q1} &= -k_t I \\ J_r \ddot{x}_{q2} &= k_t I \end{aligned} \quad (2.8)$$

2.3 The Experimental Setup

2.3.1 The Mechatronics Kit Hardware

The experimental setup that we worked on is the Mechatronics Kit from Quanser. The Pendubot, the Furuta Pendulum and the Reaction Wheel Pendulum can be assembled using the components that are provided with the kit. The angular measurements are made using relative optical encoders. These optical encoders have a resolution of 4096 divisions per 2π radians. Therefore, the smallest angle that we can measure is $2\pi/4096$. They measure each angle in 24 bits. The actuating torque is given by a 24 V brushless DC motor. Details regarding the time constants, torques and currents can be found in the datasheet.

The controller is implemented using TI's C6711 Digital Signal Processor. The motherboard of the controller is TI's C6711DSK. There is a daughter board on the motherboard which does the signal transmission required between the optical encoder and the controller on

the one hand and between the PWM card and the controller on the other. The time instants between the control actions has been programmed to be $5ms^2$. The daughter card can be used to print relevant control information on the LCD display.

This Digital Signal Processor is a floating point processor. the architecture is designed in such a way that it works in real time - an essential prerequisite for a controller³. It uses threading to accomplish the real-time capability. Of course, there is a limit to the computational load that the DSP can handle and still maintain its real-time performance. Further details can be found in the manuals accompanying the kit.

2.3.2 The Software

The CCStudio software which has been provided by TI for specific use with the DSP is used as a programming environment. The interface is very similar to the Visual Studio interface. It is a very powerful software environment allowing a whole range of real-time debugging tools.

The software development cycle consists of three main tasks. First a C program is made which has all the necessary control actions to control the hardware and all the necessary communication actions to communicate with the PC. Then this is linked, compiled and assembled into the assembly language of the DSP⁴. Finally it is downloaded into the DSP using a JTAG communication channel established through the parallel port.

A Visual Basic plug-in is written so that the PC can communicate with the RTDX libraries functioning in the DSP. This also gives an opportunity to get make a good Graphical User Interface.

2.3.3 System Parameters

The system parameter values given in manual [30] have been used through out this work. These parameter values are as follows:

$$m_p = 0.2164 \text{ kg} \quad J_p = 0.0002233 \text{ kgm}^2 \quad l_p = 0.1173 \text{ m}$$

²Indeed, the following analysis should be digital in principle, but one uses continuous time analysis for ease of mathematical tractability and because it works.

³There are libraries for Real Time Data Xchange(RTDX)

⁴Writing directly into the assembly language is a nontrivial task.

$$\begin{aligned}
m_r &= 0.0850 \text{ kg} \quad J_p = 0.00002495 \text{ kgm}^2 \quad l_p = 0.1270 \text{ m} \\
m &= 0.3014 \text{ kg} \quad J = 0.004572 \text{ kgm}^2 \quad l = 0.12 \text{ m} \\
\omega_p &= \sqrt{\frac{mgl}{I}} = 8.856 \text{ rad/sec} \\
k_t &= 0.0274 \text{ Nm/A} \quad I_{max} = 1.8 \text{ A}
\end{aligned}$$

where ω_p is the natural frequency of oscillation of pendulum.

2.3.4 Some Considerations during Experimentation

The optical encoders are relative. They have to be initialised in the zero position before running the controller. A function for this purpose has already been written and is in the file *c6xdskdigio.c* provided by the Quanser Consulting Inc. Directly we get the pendulum angle ψ_p and rotor angle ψ_r from measurement.

Another point of concern in the implementation of the control law is the measurement of disc velocity and the pendulum velocity. While it is judicious to design observers for this purpose, it is found that using a backward difference rule and then a first order filter is adequate in most control actions.

The motor torque computed by the controller undergoes a scaling as it goes through the PWM card and the counters. The control variable (u) used in the computer is scaled so that 10 units of u correspond to maximum current I_{max} . The following is the relation for control input scaling :

$$k_t I = k u; \quad |u| \leq 10$$

where proportionality constant k satisfies

$$k = \frac{k_t I_{max}}{10} = 0.00493$$

Taking u as control input the system dynamics can have following two forms:

1. With vertical downward axis as the reference axis

$$\begin{aligned}
\ddot{x}_{q1} + a \sin x_{q1} &= -b_p u \\
\ddot{x}_{q2} &= b_r u
\end{aligned} \tag{2.9}$$

2. And with vertical upward axis as the reference axis

$$\begin{aligned}\ddot{x}_{q1} - a \sin x_{q1} &= -b_p u \\ \ddot{x}_{q2} &= b_r u\end{aligned}\tag{2.10}$$

where $a = \frac{mgl}{J}$, $b_p = \frac{k}{J}$, and $b_r = \frac{k}{J_r}$. From specified system parameters these are calculated to be $a = 78$, $b_p = 1.08$ and $b_r = 198$.

Most importantly, there is a phase shift of π between the control torque and the real torque. This is the most fundamental thing about this setup that one must take care of.

2.4 Linearisation

One of the most basic techniques in studying the stability of nonlinear systems around an equilibrium point is to linearise the system dynamics around that equilibrium point. If the eigenvalues of the state matrix of the linearised system are not on the imaginary axes, conclusions about the stability of the nonlinear system around that particular equilibrium point can be drawn. In practice, most controllers for nonlinear systems whose region of operation is around the equilibrium point, are designed using linear control techniques on the linearised system.

The following sections are on the analysis of the linearised system and the design of linear controller with state feedback.

2.4.1 Analysis of Linearized Model

In the absence of the control input the system equations reduce to,

$$\begin{aligned}\ddot{x}_{q1} + a \sin x_{q1} &= 0 \\ \ddot{x}_{q2} &= 0\end{aligned}\tag{2.11}$$

The variable corresponding to the angular position of the disc does not appear in the Lagrangian and is therefore, neglected in further analysis. By using the state variables $y =$

$[y_1, y_2, y_3]^T = [x_{q1}, \dot{x}_{q1}, \dot{x}_{q2}]^T$, the state equation is written as,

$$\dot{y} = f(y)$$

where $f(y) = \begin{bmatrix} y_2 & -a \sin y_1 & 0 \end{bmatrix}^T$.

The equilibrium points are the solutions of the equation $f(y) = 0$. It is clear that the solutions are $(y_1, y_2, y_3) = (n\pi, 0, y_3)$, $n = 0, \pm 1, \pm 2, \dots$. The solutions for even values of n correspond to the downward position of the pendulum and the solutions corresponding to the odd values of n are indicators of the inverted position.

Linearising the pendulum dynamics about the downward equilibrium points we get,

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Clearly, this is an elliptic(stable) equilibrium point. The orbits close to this equilibrium point in the phase-space are ellipses. Linearising the pendulum dynamics about the vertical upward position $(y_1, y_2) \rightarrow (y_1 + \pi, y_2)$, we get

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

This is a hyperbolic (unstable) equilibrium point. The orbits close to this equilibrium point are hyperbolas. The objective of linear control design problem is to stabilize the IWP at the vertical top unstable equilibrium point.

2.4.2 The State Feedback Control Law for Linearized System

The controllability of the linearized system is checked before designing a state feedback control law.

The linearised system is given in state space model as:

$$\dot{y} = Ay + bu$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -b_p \\ b_r \end{bmatrix}$$

The condition for complete state controllability is $\text{rank}(Q) = \text{rank} \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = 3$. It can be checked that this condition is satisfied because $\det(Q) \neq 0$.

Let us consider the state feedback control law

$$u = -k_1y_1 - k_2y_2 - k_3y_3$$

where k_1, k_2 and k_3 are state feedback gains. The objective is to determine these feedback gains to meet desired specifications on system performance characteristics. During linearized control, the initial system configuration is very near the top unstable unequilibrium position which can be specified by the vector $y = (y_1(0), 0, 0)^T$. The objective of the state feedback controller is to drive the system from this initial state to final state given by $y = [0, 0, 0]^T$. The relation between final state and initial state in terms of transfer function while including state feedback control law can be shown to be as follows:

$$Y_1(s) = \frac{(s + b_r k_3)(s + b_p k_2)}{s^3 + (b_r k_3 - b_p k_2)s^2 - (a + b_p k_1)s - ab_r k_3} y_1(0) \quad (2.12)$$

Since at the top equilibrium point all states must converge to zero, $y_{1,ss} = \lim_{s \rightarrow 0} sY_1(s) = 0$, so also the disc angular velocity must converge to zero. For finding the feedback gains, following third order polynomial is selected

$$(s^2 + 2\zeta\omega s + \omega^2)(s + \alpha\omega)$$

with appropriate values of damping ζ , bandwidth ω and the real pole at $-\alpha\omega$. When compared with the following closed form characteristic polynomial,

$$A(s) = s^3 + (b_r k_3 - b_p k_2)s^2 - (a + b_p k_1)s - ab_r k_3$$

the state-feedback gains can be determined as follows:

$$\begin{aligned} k_1 &= -\frac{(a + (1 + 2\alpha\zeta)\omega^2)}{b_p} \\ k_2 &= -\frac{((\alpha + 2\zeta)\alpha\omega + \alpha\omega^3)}{ab_p} \\ k_3 &= -\frac{\alpha\omega^3}{ab_r} \end{aligned} \quad (2.13)$$

In practise, the best course is to choose a damping ζ of around 0.7 and the value of α is fixed at 0.1. Then the value of ω is slowly varied from $\omega_p = 8.856$ upwards untill a satisfactory system response is observed. The reason for choosing the third pole towards the origin rather than away from the origin is that the torque requirement in the latter case is larger.

2.4.3 Simulation and Experimental Results

Simulation and experimental results with state feedback control for linearized system are shown in Figures 2.2 and 2.3. The parameter value of ω is fixed at $\omega = 17$ to obtain satisfactory response. The simulation results show that the state feedback controller for linearized system performs well as long as the pendulum angle $|x_{q1}| \leq 0.12$. This finding was also verified during experiment. In fact this range is too small. Thus the obvious question arises if we can still employ other forms of linearized control to increase this range. What is the maximum range of the pendulum angle around top unstable equilibrium point for which the linearised control scheme can stabilize the system? The answer to this question will be investigated in the next chapter.

However, let us take a close look at simulation and experimental results obtained using this simple state feedback controller. The results are shown in figures 2.2 and 2.3. The plots of pendulum angle, angular velocity and wheel angle are compared in figure 1.2 while the plots of wheel angle velocity, control input and phase plot are compared in figure 1.3 between simulation and experiential outcomes. Following observations are made:

1. The pendulum angle exactly converges to zero in simulation while this angle has a small offset in steady state during actual experiment. Since encoder senses position, the offset might have occurred due to measurement offset.
2. The pendulum angular velocity also converges to zero during simulation, however the same state variable appears to fluctuate around zero during experiment. It should be noted that the velocity variable is not directly sensed but is estimated from sensed position using optical encoder. The differentiation of position variable might have amplified measurement noise.

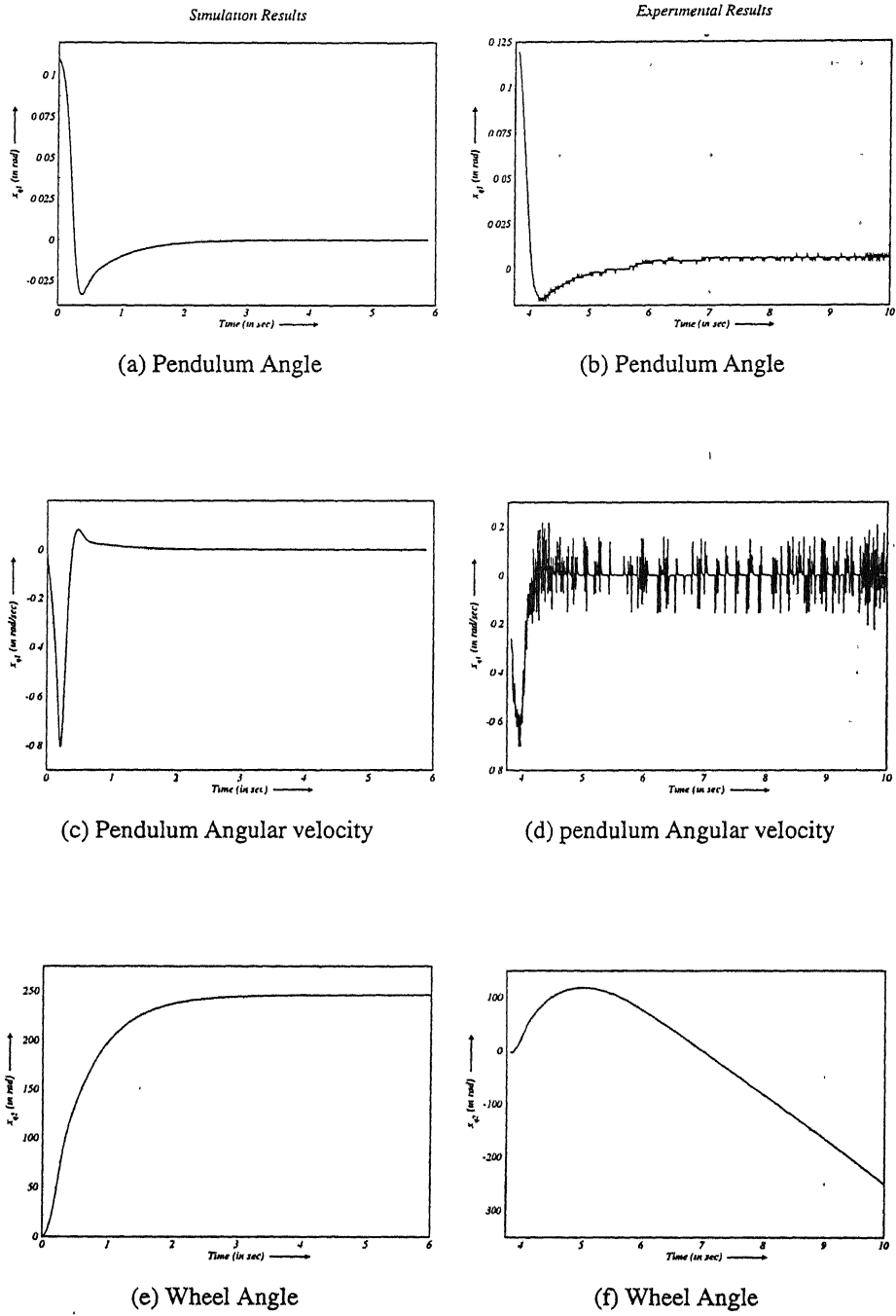


Figure 2.2: Simulation and Experimental Results For Linearised Control ($x_{q1} \sim t$, $\dot{x}_{q1} \sim t$, $x_{q2} \sim t$)

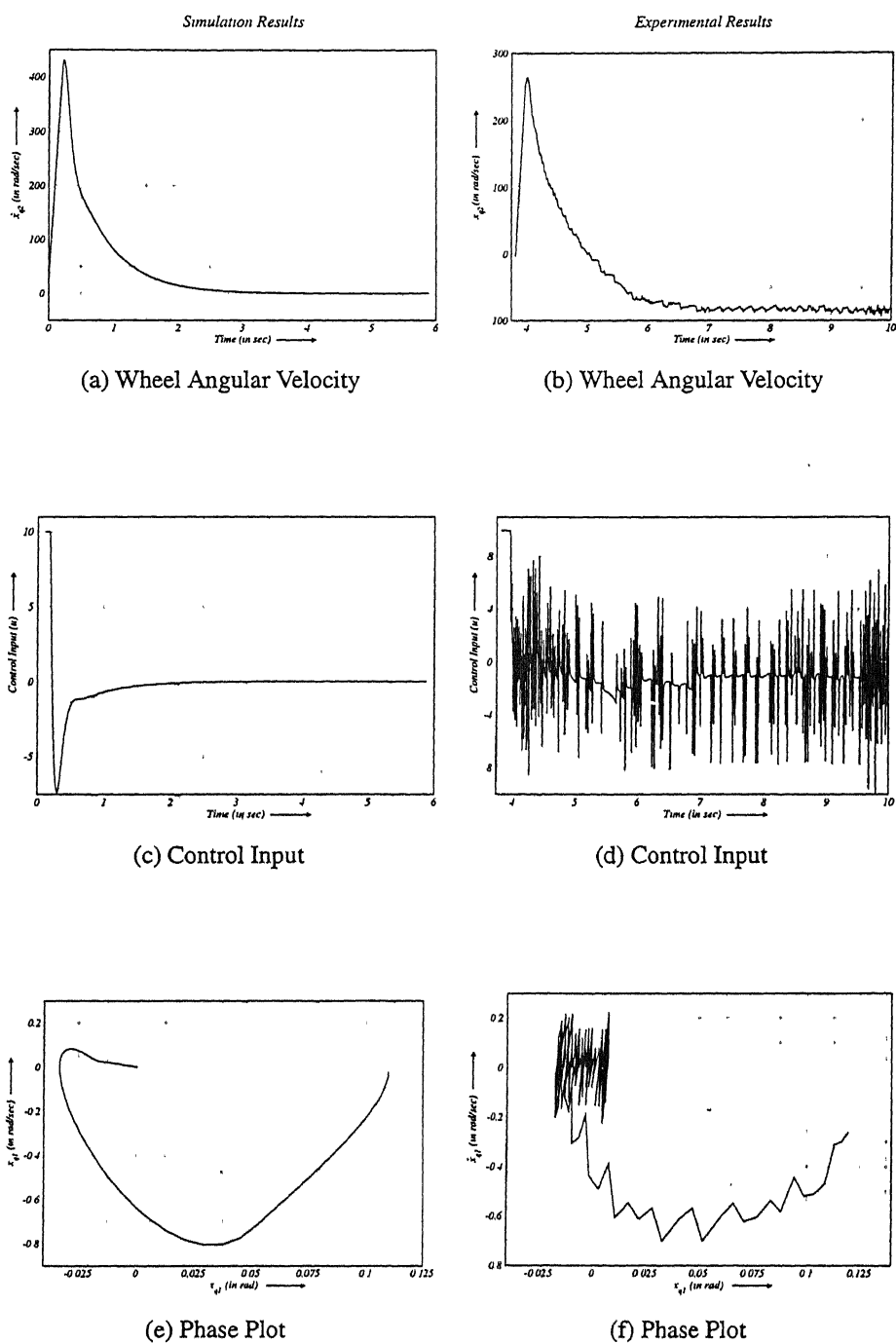


Figure 2.3: Simulation and Experimental Results For Linearised Control ($x_{q2} \sim t$, $u \sim t$, $\dot{x}_{q1} \sim x_{q1}$)

3. Although the wheel angle converges to a constant value during simulation, the same variable slowly decreases during experiment. This effect can be observed as the disc is very slowly rotating during experiment.
4. However wheel angular velocity follows same pattern during both simulation and experiment.
5. The control input is very smooth during simulation while it is fluctuating during experiment. However, the interesting aspect is that this fluctuation occurs approximately around the smooth values predicted by simulation.
6. The phase plot during simulation is very smooth while during experiment, this plot has fluctuations around the smooth trajectory predicted by simulation.

2.5 Concluding Remarks

In this chapter, the system dynamics of the Inertia Wheel Pendulum are derived using EL formulations. The system parameter values as given in experimental kit manual[30] are taken for designing various controllers during thesis work. The bonafide nature of system parameters is established by comparing simulation and experimental results while implementing a simple state feedback controller.

* * *

Chapter 3

A hybrid Control of IWP

3.1 Introduction

In this chapter a control approach for complete stabilization of Inertia Wheel Pendulum with a dual mode control is revisited. It is first shown that the system is locally feedback linearizable by a local diffeomorphism in state space and nonlinear feedback. We compare the feedback linearization control with a linear pole-placement control for the problem of balancing the pendulum about the inverted position. For the swingup problem a controller based on collocated partial feedback linearization has been discussed. As basic concept of feedback linearization some part of Differential geometry (see [2] for details) is covered in this chapter.

3.2 Feedback Linearisation

The goal of feedback linearisation is to linearise the system via feedback. It is difficult, given an arbitrary state equation, to guess a co-ordinate transformation so that the state equations become linear. In control theory, differential geometric tools have been used to find the conditions under which there is a local diffeomorphism¹. These tools also provide the procedure in finding this diffeomorphism. In case this, diffeomorphism can be made global, the system is called globally feedback linearisable. A thorough exposition of this is given by the classic books by Isidori[3],[2].

¹A function $f : U \rightarrow V$ is a diffeomorphism if it is bijective and both f and f^{-1} are continuous and have derivatives of all orders.

In this chapter, it is shown that the reaction wheel pendulum is partially feedback linearizable in a certain domain of its state-space. Spong et al.[1] showed for the first time that the reaction wheel pendulum is partially feedback linearisable in a limited region of its state space. The analysis is an application of the technique proposed in [1]. Then the simulations and experiments results are presented.

3.2.1 Mathematical Preliminaries

Before performing the feedback linearization, one requires to know some fundamental concepts of *Differential Geometry*. In this subsection some portion of differential geometry is covered. This includes *Lie derivatives*, *Lie brackets*, *Diffeomorphism*, *State transformation* and *Frobenious theorem*.

Lie Derivatives and Lie Brackets

Definition 3.1 Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth scalar function, and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth vector function on $\mathbb{R}^n$, then the Lie Derivative of h with respect to f is a vector field defined by

$$L_f h = \nabla h f$$

Thus, the Lie derivative $L_f h$ is simply the directional derivative of h along the direction of vector f . Repeated Lie derivative can be defined recursively as

$$\begin{aligned} L_f^0 h &= h \\ L_f^i h &= L_f(L_f^{i-1} h) = \nabla(L_f^{i-1} h) f \end{aligned}$$

Similarly, if g is another vector field, then scalar function $L_g L_f h(x)$ is

$$L_g L_f h = \nabla(L_f h) g$$

Definition 3.2 Let f and g be two vector fields on $\mathbb{R}^n$. The Lie Bracket of f and g is a third vector field defined by

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

The Lie bracket $[f, g]$ is commonly written as $ad_f g$. Repeated Lie brackets can then be defined recursively by

$$\begin{aligned} ad_f^0 g &= g \\ ad_f^n g &= [f, ad_f^{n-1} g] \end{aligned}$$

Diffeomorphisms and State Transformations

The concept of diffeomorphism can be viewed as a generalization of the familiar concept of coordinate transformation, according the definition stated in Slotine and Lee[4]

Definition 3.3 *A function $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, defined in a region Ω , is called a diffeomorphism if it is smooth, and if its inverse Φ^{-1} exists and is smooth.*

If the region Ω is the whole space $\mathbb{R}^n$, then $\Phi(x)$ is called a *global* diffeomorphism. if the function is defined for a finite neighbourhood of a given point, then $\Phi(x)$ is called a *local* diffeomorphism. If the Jacobian matrix $\nabla\Phi$ is non-singular in a subregion of Ω around any point, then $\Phi(x)$ defines a local diffeomorphism in the subregion. A diffeomorphism can be used to transform a nonlinear system into another nonlinear system in terms of a new set of states. Consider the dynamic system described by

$$\dot{x} = f(x) + g(x)u$$

and let a new set of states be defined by

$$z = \Phi(x)$$

differentiation of z and use of dynamic equation leads to

$$\dot{z} = \frac{\partial\Phi}{\partial x}(f(x) + g(x)u)$$

One can easily write the new state space representation as

$$\begin{aligned} \dot{z} &= \bar{f}(z) + \bar{g}(z)u \\ y &= \bar{h}(z) \end{aligned}$$

where $x = \Phi^{-1}(z)$, and the functions $\bar{f}, \bar{g}, \bar{h}$ are defined accordingly.

Frobenious Theorem

The Frobenious theorem is an important tool in treatment of feedback linearisation for non-linear systems. It provides necessary and sufficient condition for solvability of a special class of partial differential equations. The statement of the theorem is given, after giving formal definitions of complete integrability and involutivity. Detail proof of the theorem is derived in [2] and [4].

Definition 3.4 A linearly independent set of vector fields $\{f_1, f_2, \dots, f_m\}$ on $\mathbb{R}^n$ is said to be completely integrable if and only if, there exists $n-m$ scalar functions $h_1(x), h_2(x), \dots, h_{n-m}(x)$ satisfying the system of PDEs

$$\nabla h_i f_j = 0 \quad (3.1)$$

where $1 \leq i \leq n-m$, $1 \leq j \leq m$, and the gradients ∇h_i are linearly independent.

Note that with the number of vectors being m and the dimension of the associated space being n , the number of unknown scalar function h_i 's involved is $n-m$ and the number of PDEs is $m(n-m)$.

Definition 3.5 A linearly independent set of vector fields $\{f_1, f_2, \dots, f_m\}$ on $\mathbb{R}^n$ is said to be involutive if and only if, there are scalar functions $\alpha_{ijk} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$[f_i, f_j](x) = \sum_{k=1}^m \alpha_{ijk}(x) f_k(x) \quad \forall i, j \quad (3.2)$$

Involutive means that if one forms the Lie bracket of any pair of vector fields from the set $\{f_1, f_2, \dots, f_m\}$, then the resulting vector field can be expressed as a linear combination of the original set of vector fields. Note that

1. Constant vector fields are always involutive, indeed the Lie bracket of two constant vectors is simply a zero vector, which can be trivially expressed as linear combination of vector fields.
2. A set composed of single vector f is involutive. Indeed,

$$[f, f] = \left(\frac{\partial f}{\partial x}\right)f - \left(\frac{\partial f}{\partial x}\right)f = 0$$

3. From Definition 3.5, checking whether a set of vector fields $\{f_1, f_2, \dots, f_m\}$ is involutive amount to checking whether

$$\text{rank}(f_1, f_2, \dots, f_m) = \text{rank}(f_1, f_2, \dots, f_m, [f_i, f_j](x))$$

for all x and all i, j .

Now the Frobenius theorem can be stated [4] as

Theorem 3.1 *let $f_1, f_2, \dots, f_m$ be a set of linearly independent vector fields. The set is completely integrable if and only if, it is involutive.*

3.2.2 Feedback linearisation or Input-State Linearisation

In this section input state linearization for single input nonlinear systems has been discussed, which can be represented by the state equations

$$\dot{x} = f(x) + g(x)u \quad (3.3)$$

with $f(\cdot)$ and $g(\cdot)$ being smooth vector fields. The systems in this form are said to be *linear in control* or *affine* form.

Definition 3.6 *A single input nonlinear system in the affine form of (3.3), with $f(x)$, $g(x)$ being smooth vector fields on $\mathbb{R}^n$, is said to be input state linearizable or feedback linearisable if there exists a region Ω in $\mathbb{R}^n$, a diffeomorphism $T : \Omega \rightarrow \mathbb{R}^n$, and a nonlinear feedback control law*

$$u = \alpha(x) + \beta(x)\nu \quad (3.4)$$

such that the new state variables $z = T(x)$ and the new input ν satisfy a linear time invariant relation

$$\dot{z} = Az + b\nu \quad (3.5)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 1 \\ 0 & 0 & 0 & \cdot & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

From above we get the equalities

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \\ &\dots\dots\dots \\ \dot{z}_n &= \nu \end{aligned} \tag{3.6}$$

The new state z is called the linearizing state, and the control law (3.4) is called the linearizing control law. The transformed linear dynamic has its A and b vector of a special form, corresponding to a linear companion form.

Conditions for Input State Linearization

At this point a natural question comes : Are all nonlinear state equations in the form of (3.3) feedback linearizable ? If not, what is the condition for such linearization exists ? The following theorem provides answers to these questions. Only statements of the theorem is given here. The detail proof is given in [4] and [2].

Theorem 3.2 *The nonlinear system in affine form (3.3), with $f(x)$ and $g(x)$ being smooth vector fields, is input state or feedback linearizable if and only if, there exists a region Ω such that the following conditions hold :*

1. *the vector fields $\{g, ad_f g, \dots, ad_f^{n-1} g\}$ are linearly independent in Ω*
2. *the set $\{g, ad_f g, \dots, ad_f^{n-2} g\}$ is involutive in Ω*

The first condition can be interpreted as simply representing a *controllability condition* for nonlinear system (3.3) analogous to the controllability condition of a linear system. The vector field $[g, ad_f g, \dots, ad_f^{n-1} g]$ become $[b, Ab, \dots, A^{n-1}b]$ and therefore their independence satisfies the invertibility of controllability matrix. So the first condition is a general controllability condition for both linear and nonlinear systems. The second condition is trivially satisfied for linear systems (which have constant vector fields), but not generally satisfied in the nonlinear case.

How to Perform Input State or Feedback Linearization

Based on previous discussion the feedback linearization of a nonlinear system of form (3.3) can be performed through following steps :

1. Construct the vector field $\{g, ad_f g, \dots, ad_f^{n-1} g\}$ for the given system.
2. Check whether controllability and involutivity conditions are satisfied.
3. If both are satisfied, from definition 3.6 we get, $z_i = T_i(x)$ where T_i denotes the i -th component of vector valued function $T(x)$. From the equalities (3.6) we get

$$\dot{z}_1 = \nabla T_1 \dot{x} = L_f T_1 + L_g T_1 = z_2 = T_2(x)$$

As $T_2(x)$ is independent of u , following is true

$$T_2(x) = L_f T_1 \text{ and } L_g T_1 = 0$$

Repeating the reasoning for $z_2, z_3, \dots, z_n$, shows that

$$L_f T_i = T_{i+1}, \quad L_g T_i = 0, \quad \text{for } i = 1, \dots, n-1 \quad (3.7)$$

Now let us show that the above equation on the T_i 's can be composed into a set of constraint equations on T_1 alone. To begin with let's consider the term $\nabla T_1[f, g]$, equation (3.7) implies that

$$\nabla T_1[f, g] = \nabla(L_g T_1)f - \nabla(L_f T_1)g = 0 - L_g T_2 = 0$$

By proceeding inductively we can show,

$$\nabla T_1 \text{ad}_f^k g = 0 \quad \text{for } k = 0, 1, 2, \dots, n-2 \quad (3.8)$$

Finally for $i = n$ we have

$$\dot{z}_n = L_f T_n + u L_g T_n = \nu \quad (3.9)$$

Which says $L_g T_n \neq 0$ due to dependency of the above equation on input u , which gives

$$\nabla T_1 \text{ad}_f^{n-1} g \neq 0 \quad (3.10)$$

Hence

$$-\frac{L_f T_n}{L_g T_n} = \alpha(x) \text{ and } \frac{1}{L_g T_n} = \beta(x)$$

3.2.3 Feedback linearisation of IWP

To perform the feedback linearization on Inertia Wheel Pendulum, consider the dynamic model of the form

$$\begin{aligned} J\ddot{x}_{q1} &= mgl \sin x_{q1} - ku \\ J_r\ddot{x}_{q2} &= ku \end{aligned} \quad (3.11)$$

where $x_{q1} = \psi_p$ and $x_{q2} = \psi_p + \psi_r$, ψ_p and ψ_r are pendulum angle and rotor angle respectively with reference to the vertical upward position of pendulum. u is the input to PWM driver and k is a proportionality constant. Taking $x_1 = x_{q1}$, $x_2 = \dot{x}_{q1}$ and $x_3 = \dot{x}_{q2}$ the equation (3.11) takes the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ a \sin x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -b_p \\ b_r \end{bmatrix} u \quad (3.12)$$

where $a = \frac{mgl}{J}$, $b_p = \frac{k}{J}$ and $b_r = \frac{k}{J_r}$. As the rotor angle position does not have importance, we have omitted it from the equation (3.12). Equation (3.12) is in affine form of relative degree $n = 3$ with

$$f(x) = \begin{bmatrix} x_2 \\ a \sin x_1 \\ 0 \end{bmatrix} \text{ and } g(x) = \begin{bmatrix} 0 \\ -b_p \\ b_r \end{bmatrix}$$

Following steps are followed to perform feedback linearization and to find out the control input for Inertia Wheel pendulum system.

STEP 1 : For the reaction wheel pendulum, the vector fields $\{g, ad_f(g), ad_f^2(g)\}$ are given by

$$g = \begin{bmatrix} 0 \\ -b_p \\ b_r \end{bmatrix}$$

$$ad_f(g) = \begin{bmatrix} ab_p \cos x_1 \\ 0 \\ 0 \end{bmatrix}$$

$$ad_f^2(g) = \begin{bmatrix} -ab_p x_2 \sin x_1 \\ -ab_p \cos x_1 \\ 0 \end{bmatrix}$$

STEP 2 : Since $\det \begin{bmatrix} f & ad_f(g) & ad_f^2(g) \end{bmatrix} = -ab_r b_p^2 \cos^2 x_1$ this set of vector fields is linearly independent in the region $|x_1| < \frac{\pi}{2}$. To test involutivity, observe that,

$$[g, ad_f(g)] = \frac{\partial ad_f(g)}{\partial x} g = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which is a linear combination of the original vector fields.

STEP 3 : To find out the transformation we follow the step 3 said in last subsection, so conditions imply

$$\begin{aligned} \frac{\partial T_1}{\partial x_1} &= 0 \\ b_p \frac{\partial T_1}{\partial x_2} &= b_r \frac{\partial T_1}{\partial x_3} \\ \frac{\partial T_1}{\partial x_2} &\neq 0 \end{aligned}$$

The simple choice $T_1 = b_r x_2 + b_p x_3$ satisfies the above three conditions. The other two components of the transformation can be computed as

$$\begin{aligned} T_2 &= L_f T_1 = ab_r \sin x_1 \\ T_3 &= L_f T_2 = ab_r x_2 \cos x_1 \end{aligned}$$

Finally we get control input

$$u = -\frac{\nu - a^2 b_r \cos x_1 \sin x_1 + ab_r x_2^2 \sin x_1}{ab_p b_r \cos x_1} \quad (3.13)$$

So defining new coordinate $z = T(x)$ i.e.

$$\begin{aligned} z_1 &= b_r x_2 + b_p x_3 \\ z_2 &= ab_r \sin x_1 \\ z_3 &= ab_r x_2 \cos x_1 \end{aligned} \quad (3.14)$$

the system equations (3.11) takes the linear form of equation(3.5).

3.2.4 Control Design

As it is seen in the last section, the diffeomorphism $T(x)$ transforms the system equations to the following form:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= \nu \end{aligned}$$

This is valid for $|x_1| = |x_{q1}| < \pi/2^2$. Now the linear state feedback technique can be used to get the following control law,

$$\nu = -k_1 z_1 - k_2 z_2 - k_3 z_3 \quad (3.15)$$

²This kind of feedback linearization is also applicable about the downward equilibrium position.

where $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}^T$ is called the feedback gain matrix. Again, one compares the characteristic equation with the standard third order polynomial

$$(s + \alpha\omega)(s^2 + 2\zeta\omega s + \omega^2)$$

to get the values of (k_1, k_2, k_3) for the desired controller response. Note that, this guarantees $(z_1, z_2, z_3) \rightarrow 0$ which in turn can be seen to indicate $(x_1, x_2, x_3) \rightarrow 0$.

3.2.5 Simulation and Experimental Results for Feedback Linearization

The simulation and experimental results are shown in figures 3.1 and 3.2. The parameter values of ω , α , ζ are fixed at $12.0rad$, 0.2 and 0.6 respectively. The simulation results shows that, for pendulum angle and pendulum angular velocity within range $|x_{q1}| \leq 0.12$ and $|\dot{x}_{q1}| \leq 2.0$, the balancing controller works well, this is also observed with experiment. So the above ranges specify the domain of attraction of the vertical upward position with balancing controller.

Now let us give a close look at the simulation and experimental results for balancing controller making use of feedback linearisation. The plots of pendulum angle, pendulum angular velocity and wheel angle are compared in figure 3.1 while the plots of wheel angular velocity, control input and phase plot are compared in figure 3.2 between simulation and experimental results. The observations are summarised as below :

1. The pendulum angle converges to zero in simulation while the angle has a small steady state offset during actual experiment. This offset is due to encoder measurement offset.
2. The pendulum angular velocity obtained by both simulation and experiment converges to zero at steady state. But the pendulum angular velocity obtained from experiment has some oscillations. As the angular velocity is estimated from sensed angular position by optical encoder, the differentiation of position variable amplifies the measured noise, which appears as oscillations in pendulum angular velocity.
3. In simulation the disc angle finally reaches to a steady state value, but in experiment due to presence of small roatation it continuously goes on increasing.

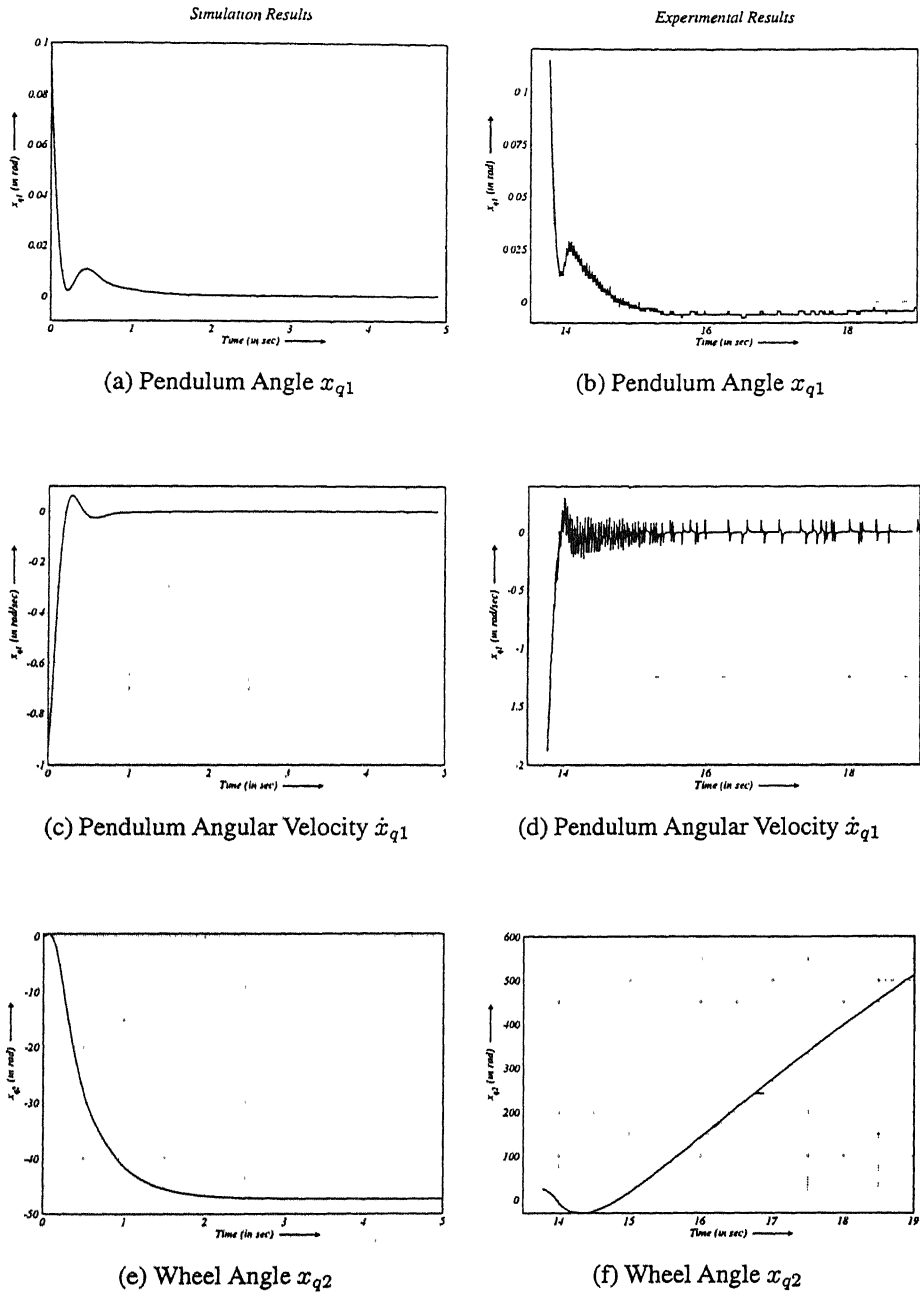
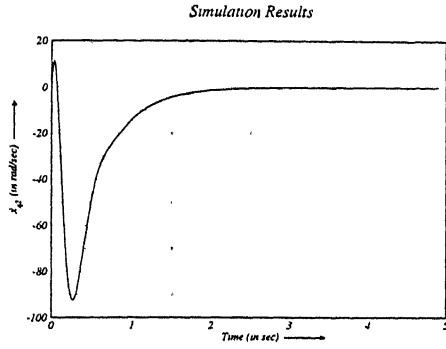
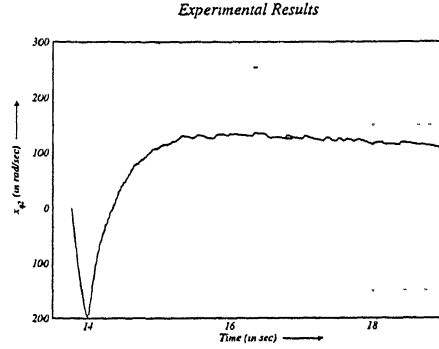


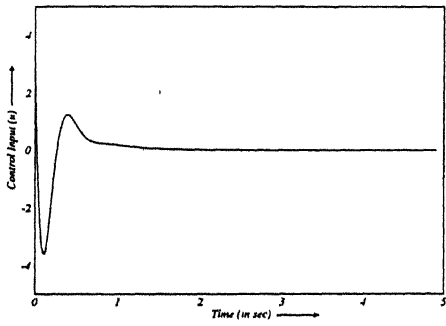
Figure 3.1: Simulation and Experimental Results of Feedback Linearization ($x_{q1} \sim t$, $\dot{x}_{q1} \sim t$, $x_{q2} \sim t$)



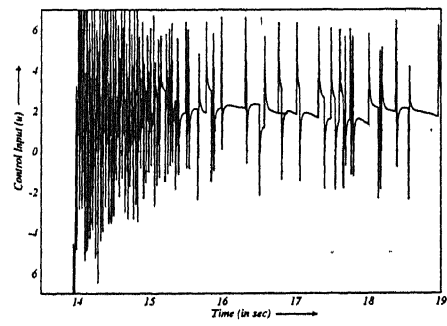
(a) Wheel Angular Velocity $\dot{x}_{q2}$



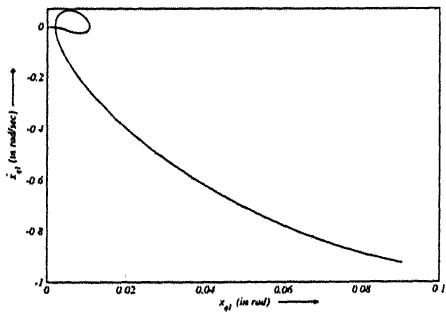
(b) Wheel Angular Velocity $\dot{x}_{q2}$



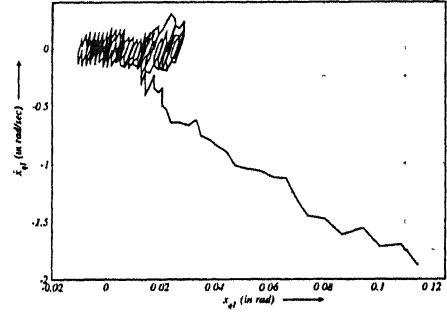
(c) Control Input u



(d) Control Input u



(e) Phase Plot $\dot{x}_{q1} \sim x_{q1}$



(f) Phase Plot $\dot{x}_{q1} \sim x_{q1}$

Figure 3.2: Simulation and Experimental Results of Feedback Linearization ($x_{q2} \sim t$, $u \sim t$, $\dot{x}_{q1} \sim x_{q1}$)

4. The disk velocity decreases to zero in simulation, but some velocity remains in experimental results.
5. The control input is very smooth in case of simulation, but in experiment it is too much noisy. As we are estimating the input from state variables, presence of noise in the measured and estimated states might be the cause of presence of noise in input.
6. Phase plot obtained in experiment is nearly identical with the result obtained in simulation except a small disturbance is present.

3.3 Partial Feedback linearization

In this section a technique used by Spong[31] (to control the Acrobot) is used to control the Reaction Wheel Pendulum. The object of this strategy is to swing up the pendulum from any position to a point close to the inverted position. Note that it does not balance the pendulum at the inverted position. However, it is possible to switch to a balancing controller once it reaches into the region where balancing controller works..

3.3.1 Analysis

The basic idea is to *pump energy* into the system by rotating the disk *in phase* with the motion of the pendulum so that the amplitude of the pendulum increases with each swing. This is called the *collocated partial feedback linearisation* strategy in [31] because the actuated joint is linearised. Consider the following equations of the reaction wheel pendulum

$$\begin{aligned}\ddot{x}_1 - a \sin x_1 &= -b_p u \\ \ddot{x}_2 &= b_r u\end{aligned}\tag{3.16}$$

Where the reference for pendulum angle ψ_p and wheel angle ψ_r is the vertical downward axis. The second equation is just a double integrator and can be linearised by the following choice of control law,

$$u = \ddot{x}_{2d} - k_2(\dot{x}_2 - \dot{x}_{2d}) - k_1(x_2 - x_{2d})\tag{3.17}$$

where x_{2d} is the reference trajectory. It was shown in [32] that an appropriate choice of the reference trajectory can be made so that the resulting zero dynamics³ are unstable. This causes the pendulum to swing to a position near the inverted poition.

The choice of the reference trajectory which causes such a wonderful thing to happen is

$$x_{2d} = -k_3\sigma(\dot{x}_1) \quad (3.18)$$

where $\sigma(\cdot)$ is a sigmoidal function. A heuristic explanation of the reason for this based on energy arguments is given in [31], while a more rigorous explanation based on the analysis of the resulting zero dynamics is presented in [32],

3.3.2 Control Design

We have four design parameters, namely (k_1, k_2, k_3) and the choice of the function $\sigma(\cdot)$. Unlike in the case of the Acrobot where teh position of the second link is a matter of grave concern and the choice of a smooth sigmoidal function has its benefits, we opt for an abrupt nonlinearity, i.e.

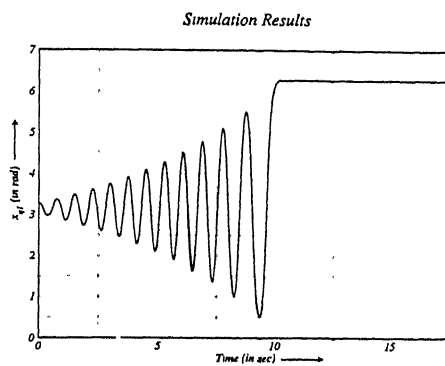
$$\sigma(\dot{x}_1) = \text{sgn}(\dot{x}_1) \quad (3.19)$$

The parameters (k_1, k_2) can be assigned for a good enough second order controller response. The values of k_3 and the speed of the controller response have to be determined by looking at the response of the real system. Roughly, one wishes to increase the energy transfer between the links so we wish that the second link reach its reference trajectory in a leisurely way.

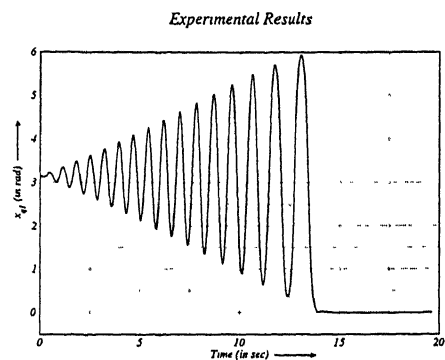
3.3.3 Simulation and Experimental Results of Complete Stabilization

We combine this swingup routine with the linear balancing controller to present a hybrid controller strategy, which succeeds in swinging up th pendulum from the downward position and stabilising it in the inverted position. The choice of the angle where the switching takes place is determined experimentally to be 0.11 rad in either direction. For the partial fee back linearization controller, the parameter values of ω and ζ are fixed at 15.0 and 0.7 respectively.

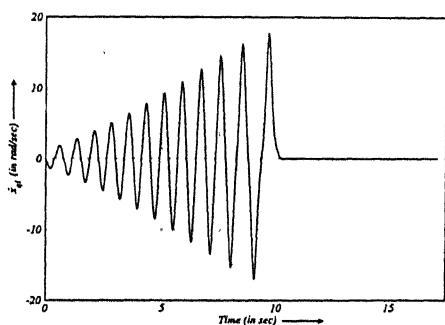
³The evolution of the unlinearized link(the pendulum angle).



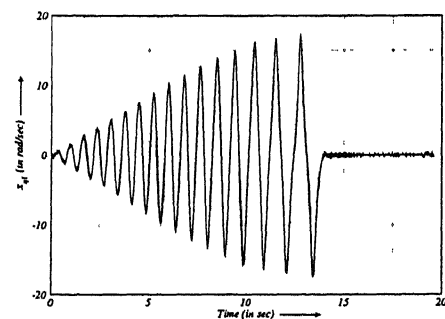
(a) Pendulum Angle x_{q1}



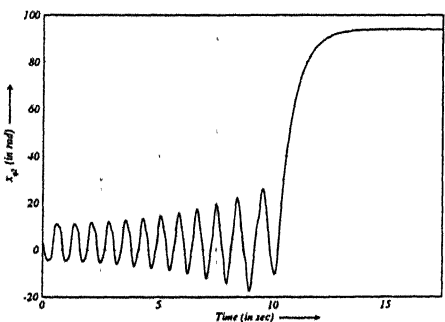
(b) Pendulum Angle x_{q1}



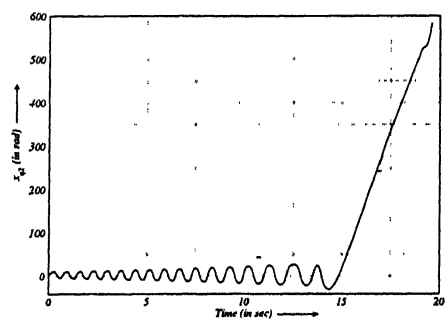
(c) Pendulum Angular Velocity $\dot{x}_{q1}$



(d) Pendulum Angular Velocity $\dot{x}_{q1}$

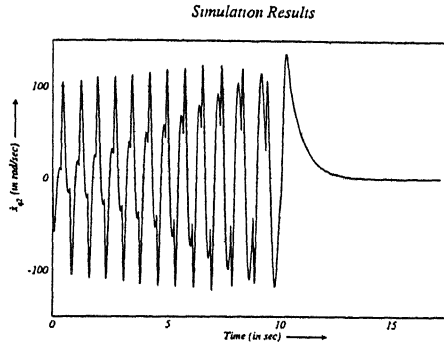


(e) Wheel Angle x_{q2}

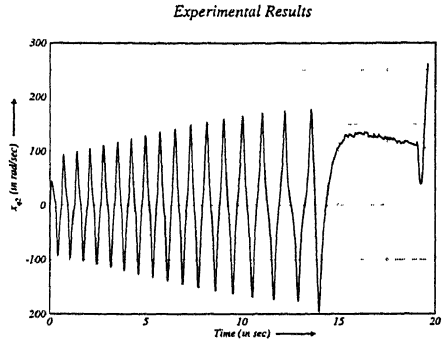


(f) Wheel Angle x_{q2}

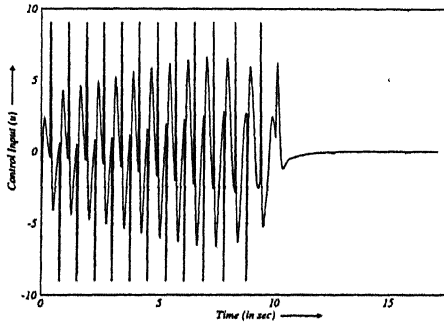
Figure 3.3: Simulation and Experimental Results of Complete Stabilisation ($x_{q1} \sim t$, $\dot{x}_{q1} \sim t$, $x_{q2} \sim t$)



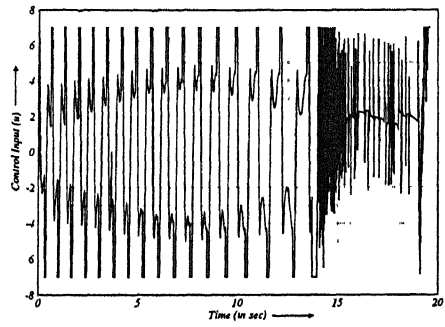
(a) Wheel Angular Velocity $\dot{x}_{q2}$



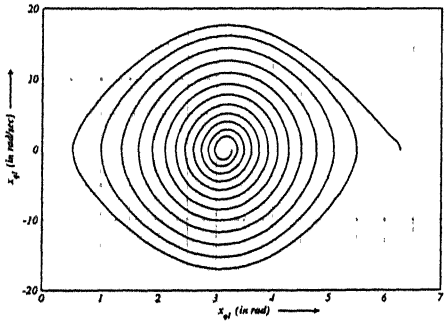
(b) Wheel Angular Velocity $\dot{x}_{q2}$



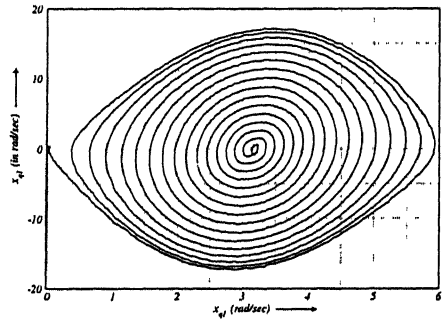
(c) Control Input u



(d) Control Input u



(e) Phase Plot $\dot{x}_{q1} \sim x_{q1}$



(f) Phase Plot $\dot{x}_{q1} \sim x_{q1}$

Figure 3.4: Simulation and Experimental Results of Complete Stabilisation ($x_{q2} \sim t$, $u \sim t$, $\dot{x}_{q1} \sim x_{q1}$)

From both simulation and experiment, it is determined that when pendulum angle reaches within the range of $\pm 0.12 \text{ rad}$ the pendulum angular velocity remain well within the range of ± 2.0 , hence switching over to balancing control stabilizes the pendulum at unstable equilibrium point. Simulation and experiment results for complete stabilization are compared in figure 3.3 and 3.4. Here we give a close look to the simulation and experimental results.

1. The figure 3.3 compares the simulation and experimental results of pendulum angle, Pendulum angular velocity and wheel angle.
2. It is observed from figure 3.3 that, in simulation the pendulum takes less time to reach vertical upward position from downward position, where as in experiment it takes a longer time. Perhaps this time difference is due to presence of friction which we are neglecting in simulation. In experiment we are injecting energy into the system to lift the pendulum up, if some part of it losses due to friction, then obviously it will take longer time than expected without friction.
3. Figure 3.4 gives a comparison between simulation and experimental results for Wheel angular velocity, control input and phase plot.
4. In simulation it is observed that the disc velocity finally goes to zero at steady state, but in experiment there remains a small velocity. Possibly this is due to presence of small torque exerted by the dc motor supply wire. To neutralise the wire torque the disk rotates at a small speed. This is the cause of continuous increase in wheel angle.
5. Control input responses for both simulation and experiment are almost identical during swing up of pendulum, but some amount of noise is present during stabilization.

3.4 Concluding Remarks

In this chapter we revisited the hybrid nonlinear control of a Inertia Wheel Pendulum to achieve both swingup and balancing. We investigate the balancing control strategy based on feedback linearization and swing up based on partial feedback linearization. We have shown that the system is locally feedback linearizable by a local diffeomorphism (*i.e.* in

region $|x_{q1}| < \frac{\pi}{2}$) in state space and nonlinear feedback. In practice the performance of feedback linearization control is comparable to approximate linearization and pole placement control discussed in Chapter 2. The performance of the system is limited by finite actuator torque capability. The swingup control we discussed is based on collocated partial feedback linearization.

* * *

Chapter 4

Passivity Based Control of IWP

4.1 Introduction

Nonlinear systems has undergone tremendous development over the last three decades. Introduction of many techniques has greatly advanced the theory and has enabled the generalization of many fundamental concepts for linear systems to the nonlinear world. In last decade the theory of passive systems has brought a revolution in this area. With this theory the latest developments on possibility of shaping the physical energy in such a way that it can be used as a suitable Lyapunov function for the control purpose has led to the control technique what is called *Passivity-based Control*. Passivity based control (PBC) is a well established structure preserving design methodology which has been shown to be very powerful to design robust controllers for physical systems described by Euler Lagrangian (EL) equations of motion. For regulation problems of mechanical systems it provides a natural procedure to *shape* the *potential energy* yielding controllers with a clear physical interpretation.

4.2 Dissipativity and passivity

The properties of dissipativity and passivity are very important in the analysis of EL systems. The theory presented is mostly based on passive systems. A passive system is a system that can not store more energy than the energy supplied to it. In other words, a passive system

does not contain an energy source. Consider the non-linear system

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + g(x(t))u(t) \\ y(t) &= h(x(t)) + j(x(t))u(t)\end{aligned}\tag{4.1}$$

where $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}^m$, and the nonlinear functions, defined by $f(\cdot), g(\cdot), h(\cdot), j(\cdot)$, are smooth. Let us call $S(u(t), y(t))$ the supply rate, such that $\forall u, \forall x(0)$

$$\int_0^t |S(u(t), y(t))| dt < \infty \quad t \in \mathbb{R}^+$$

Definition 4.1 (*Dissipativity*) An EL system is dissipative with respect to the supply rate $S(u, y)$ if and only if there exists a storage function H such that

$$H(x(t)) \leq H(x(0)) + \int_0^t S(u(t), y(t)) dt\tag{4.2}$$

$\forall u$ and $\forall x(0)$.

This is called dissipation inequality, which means the storage energy function $H(x(t))$ at a future time t is not bigger than the sum of the available energy function $H(x(0))$ at an initial time 0 plus the total energy $\int_0^t S(u(t), y(t)) dt$ supplied to the system from external source in the interval $[0, t]$. There is no internal creation of energy.

Passive system represent an important subset of dissipative systems.

Definition 4.2 (*Passivity*) A system is passive if it is dissipative with supply rate $S(u, y) = u^T y$. Because the supplied energy is greater than or equal to the total energy H , the passivity property can also be written as

$$H(x(T)) - H(x(0)) \leq \int_0^T u^T(t) y(t) dt\tag{4.3}$$

$\forall u(t), \forall T \geq 0$ and all $H(x(0))$

Definition 4.3 The system given in equation 4.1 is passive if there exists a (locally) positive definite scalar function H , called a storage function, such that

$$\dot{H} \leq u^T y\tag{4.4}$$

Integration of equation 4.4 leads to same condition as given in definition 4.2 or equation 4.3.

4.2.1 Passive systems and their properties

Consider a mass-spring-damper system with equation of motion

$$M\ddot{x} + B\dot{x} + Kx = u$$

where M is the mass, B is the damping coefficient and K is spring constant, The total energy of the system is given as

$$H = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}Kx^2$$

Then $\dot{H}$ satisfies

$$\begin{aligned}\dot{H} &= M\dot{x}\ddot{x} + Kx\dot{x} \\ &= \dot{x}u - B\dot{x}^2 \leq \dot{x}u\end{aligned}$$

Therefore mass-spring-damper system is passive if one takes force u as the input and velocity $\dot{x}$ as the output (equation 4.4), but the system ceases to be passive if one takes position x as output since one can not verify the relation $\dot{H} < xu$. Let us revisit some of the properties of passive systems:

1. A linear system is passive if the transfer function that relates its output with input has relative degree one.
2. A passive system can be easily stabilized by proportional feedback of passive output as shown in Fig (4.1). Since the system G is passive, the storage function H follows the relation $\dot{H} < u^T y$. Because of proportional feedback, setting $u = -ky$, the rate derivative of the storage function H satisfies the following relation:

$$\dot{H} \leq u^T y = -ky^T y \leq 0$$

for any positive value of gain k . This implies that H is a decreasing function, i.e. the function H will continuously decrease until output y is not zero. Using LaSalle's theorem [24], one can show that the system is asymptotically stable.

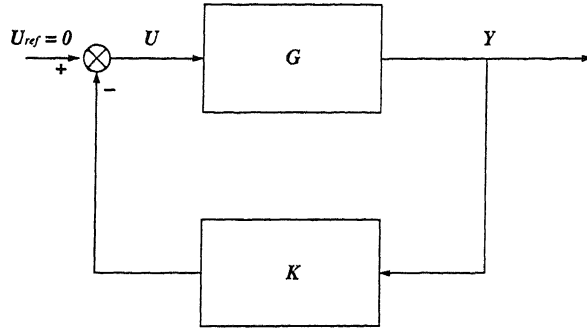


Figure 4.1: Output feedback passive system

3. The proportional gain introduced for a passive system is normally referred as *infinite gain margin*. This is because the system is always stable even if the gain is very large as long as it is positive.
4. If passive systems are connected in parallel, the resulting system is also passive as shown in Figure 4.2. With reference to figure 4.2, let H_1 and H_2 are the stored energy

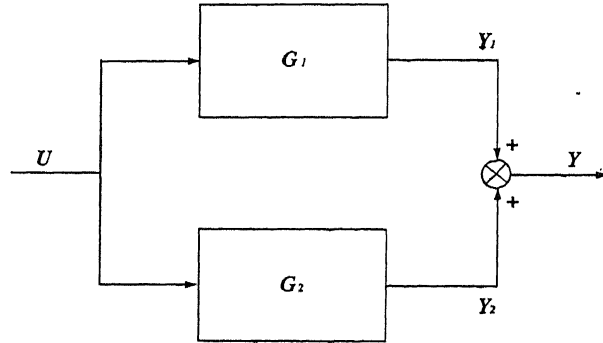


Figure 4.2: Parallel interconnection systems

function corresponding to respective subsystems G_1 and G_2 . Let y_1 and y_2 are passive outputs of these two sub-systems respectively while u is the input to each subsystem. Since both systems are passive then following relation holds good:

$$\dot{H}_1 \leq u^T y_1$$

$$\dot{H}_2 \leq u^T y_2$$

For the combined parallel interconnected system, the stored energy is $H_1 + H_2$. Thus the following relation also holds good:

$$\begin{aligned}\dot{H}_1 + \dot{H}_2 &\leq u^T y_1 + u^T y_2 \\ \Rightarrow \dot{H}_1 + \dot{H}_2 &\leq u^T (y_1 + y_2)\end{aligned}$$

Since $y_1 + y_2 = y$, the output of the combined system, the combined system is also passive.

5. If passive systems are connected in feedback, then the resulting system is also passive as shown in figure 4.3. Let two systems G_1 and G_2 are associated with stored energy

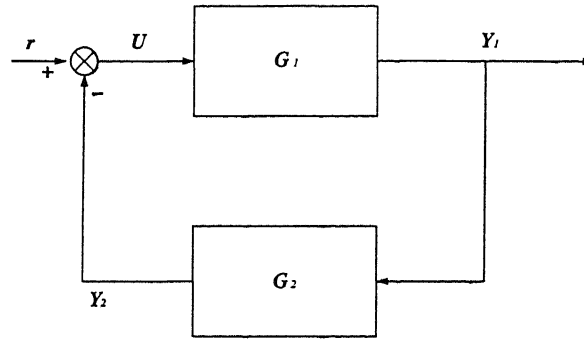


Figure 4.3: Feedback interconnection systems

functions H_1 and H_2 respectively. The following conditions are true for individual systems:

$$\begin{aligned}\dot{H}_1 &\leq u^T y_1 \\ &\leq (r - y_2)^T y_1 \\ \dot{H}_2 &\leq y_1^T y_2\end{aligned}$$

Since the stored energy function of the combined system is $= H_1 + H_2$, one can easily

obtain following relation:

$$\begin{aligned}\dot{H}_1 + \dot{H}_2 &\leq (r - y_2)^T y_1 + y_1^T y_2 \\ &\leq r^T y_1\end{aligned}$$

where r is the input and y_1 the output of the combined system. Thus it is clear that the feedback interconnection of passive systems is also passive.

4.3 Passivity Based Control (PBC) of IWP

The system dynamics of the Inertia Wheel Pendulum system as described in chapter 2 is reintroduced here:

$$\begin{aligned}J\ddot{x}_{q1} + mgl \sin x_{q1} &= -ku \\ J_r\ddot{x}_{q2} &= ku\end{aligned}\tag{4.5}$$

where $x_{q1} = \psi_p$ and $x_{q2} = \psi_p + \psi_r$, ψ_p and ψ_r are pendulum angle and rotor angle respectively measured from vertically downward axis. J and J_r are inertia parameters. u is the input to PWM driver and k is a proportionality constant.

The inertia wheel pendulum system can be thought of as an parallel interconnection of two subsystems *i.e.* pendulum subsystem and wheel subsystem with input u as shown in Figure (4.4). Then the control problem is to determine $u = f(y_1, y_2)$ such that the parallel interconnection system as shown in figure 4.4 is passive.

Before one can determine the control law $u = f(y_1, y_2)$, one needs to establish that each subsystem shown in figure 4.4 is passive. Consider the subsystem wheel whose transfer function is $G_1 = \frac{1}{s^2}$. The energy storage function H_1 for the wheel is very simple to find out which is $H_1 = \frac{1}{2}J_r\dot{x}_{q2}^2$. The rate derivative of this storage function can be simplified using the relation $J_r\ddot{x}_{q2} = ku$ given in equation 4.5:

$$\dot{H}_1 = J_r\dot{x}_{q2}\ddot{x}_{q2} = ku\dot{x}_{q2}\tag{4.6}$$

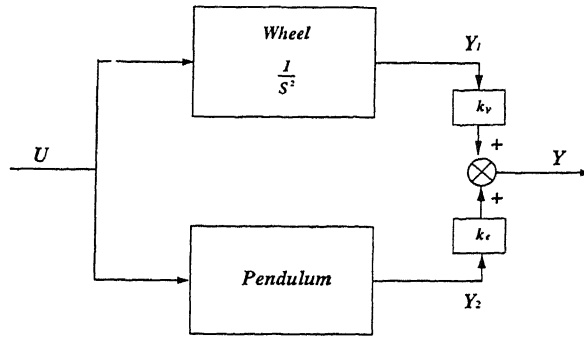


Figure 4.4: IWP system as a parallel interconnection

If one selects $\dot{x}_{q2}$ as the output y_1 of the wheel sub-system, then this subsystem is passive as shown in equation 4.6.

Next consider the second subsystem which is the pendulum system. The general form of energy function of this subsystem is given as follows:

$$E = \frac{1}{2} J \dot{x}_{q1}^2 + mgl(1 - \cos x_{q1}) \quad (4.7)$$

The rate derivative of this pendulum energy follows:

$$\begin{aligned} \dot{E} &= J \dot{x}_{q1} \ddot{x}_{q1} + mgl \sin(x_{q1}) \dot{x}_{q1} \\ &= (J \ddot{x}_{q1} + mgl \sin(x_{q1})) \dot{x}_{q1} \\ &= -ku \dot{x}_{q1} \end{aligned} \quad (4.8)$$

It can be seen in equation 4.8 that the energy is constant along the state trajectories if $u = 0$. In addition, the pendulum subsystem is passive from input $-ku$ to $\dot{x}_{q1}$, with the energy E as the storage function. However instead of taking the energy E as the stored energy function, the stored energy function is taken as follows, the motivation for which will be clear shortly:

$$H_2 = \frac{1}{2} (E - E_{ref})^2 \quad (4.9)$$

where E_{ref} is a (constant) reference energy. The rate derivative of this new stored energy

function yields following relation:

$$\begin{aligned}
\dot{H}_2 &= (E - E_{ref})\dot{E} \\
&= -k\dot{x}_{q1}(E - E_{ref})u \\
&= y_2u
\end{aligned} \tag{4.10}$$

where the output function is taken as $y_2 = -k\dot{x}_{q1}(E - E_{ref})$. The reason for such choice of y_2 and H_2 will be explained shortly. Since the pendulum system is passive as per equation 4.10 from input u to output y_2 , the parallel interconnection system of pendulum and wheel is also passive.

It is verified that the combined wheel and pendulum system is passive from input u to output $y_1 + y_2$ using property (4) in section 4.2.1. Instead of taking the output of combined system to be $y_1 + y_2$, it is taken as $y = k_v y_1 + k_e y_2$ to add extra degree of freedom to control design through proper selection of parameters k_v and k_e . The modified storage function for the combined system thus becomes:

$$\begin{aligned}
H &= k_v H_1 + k_e H_2 \\
&= \frac{1}{2}k_v J_r \dot{x}_{q2}^2 + \frac{1}{2}k_e (E - E_{ref})^2
\end{aligned} \tag{4.11}$$

Computing $\dot{H}$ along the state trajectories of the combined system yields:

$$\dot{H} = (k_v k \dot{x}_{q2} - k_e k (E - E_{ref}) \dot{x}_{q1})u = yu \tag{4.12}$$

If one selects control input as follows:

$$u = -k_u y = -k_u (k_v k \dot{x}_{q2} - k_e k (E - E_{ref}) \dot{x}_{q1}) \tag{4.13}$$

then equation 4.12 is further simplified as:

$$\dot{H} = -k_u y^2 \leq 0 \tag{4.14}$$

Thus the system is stable as per equation 4.14 using Lasalle's invariance theorem. and now we can determine the asymptotic behavior of the system making use of LaSalle's invariance principle. Setting $y \equiv 0$ to make $\dot{H} = 0$ yields

$$u = k_u(k_e k(E - E_{ref})\dot{x}_{q1} - k_v k\dot{x}_{q2}) \equiv 0 \quad (4.15)$$

It follows that the time derivative $\dot{u} \equiv 0$ from which we get

$$k_u(k_e k\dot{E}\dot{x}_{q1} + k_e k(E - E_{ref})\ddot{x}_{q1} - k_v k\ddot{x}_{q2}) \equiv 0$$

Since $u = 0$, $\ddot{x}_{q2} = \frac{ku}{J_r}$, $\dot{E} = -ku\dot{x}_{q1}$, and $\ddot{x}_{q1} = -\frac{mgl}{J} \sin x_{q1} - ku$, the above equation reduces to

$$(E - E_{ref}) \sin x_{q1} = 0 \quad (4.16)$$

This equation is very important and says that the closed loop trajectory will converge to either $E = E_{ref}$ or $\sin x_{q1} = 0$. From the first case $E = E_{ref}$ in addition to equation (4.13) we get $\dot{x}_{q2} = 0$. In second case, it follows that $x_{q1} = n\pi$ and $\dot{x}_{q2} = \text{constant}$. One thing to be noted down here, this PBC only swings up the pendulum up to vertical upward position but not stabilizes there, because the controller brings the trajectory into a homoclinic orbit which intersects both stable and unstable equilibrium points.

An extremely important advantage of the PBC is that the analysis remain unchanged if, instead of using control law defined by (4.11), we use saturated control

$$u = \text{sat}(-k_u y) = \text{sat}(-k_u(k_v k\dot{x}_{q2} - k_e k(E - E_{ref})\dot{x}_{q1})) \quad (4.17)$$

where $\text{sat}(\cdot)$ is a saturation function. Then the storage function H satisfies the condition

$$\dot{H} = -y \text{sat}(k_u y) \leq 0$$

Hence the system is stable with saturation control.

4.3.1 Effect of friction on Controller

The friction has a significant effect on the performance of the controller which can be seen from simulation and experimental results in Figure 4.5 and 4.6. Considering the friction at pendulum hinge point and neglecting the wheel friction (as it can be compensated by adding equivalent amount of motor input) the equations 4.5 becomes

$$\begin{aligned} J\ddot{x}_{q1} + b\dot{x}_{q1} + mgl \sin x_{q1} &= -ku \\ J_r\ddot{x}_{q2} &= ku \end{aligned} \quad (4.18)$$

where b is the coefficient of rotational friction at the hinge point of pendulum. We keep the energy function H , output y and input u functions of subsystems same as before. As the stored energy of the whole system is $H = \frac{1}{2}k_v J_r \dot{x}_{q2}^2 + \frac{1}{2}k_e (E - E_{ref})^2$, then $\dot{H}$ along the trajectory is

$$\begin{aligned} \dot{H} &= (k_v k \dot{x}_{q2} - k_e k (E - E_{ref}) \dot{x}_{q1})u - b(E - E_{ref})\dot{x}_{q1}^2 \\ &= yu - b(E - E_{ref})\dot{x}_{q1}^2 \\ &= -k_u y^2 - b(E - E_{ref})\dot{x}_{q1}^2 \end{aligned} \quad (4.19)$$

Now we can analyse the effect of friction by superimposing it on the ideal model. The stored energy decreases or remain constant if $\dot{H} \leq 0$, except at the vertical upward position the term $-b(E - E_{ref})\dot{x}_{q1}^2$ is always positive (as $E \leq E_{ref}$ and $\dot{x}_{q1}$ is non zero), and the term $-k_u y^2$ is negative (Except at the vertically upward position of pendulum where it is zero) and gradually approaches zero as the pendulum approaches vertical upward position, so there is some region around the vertical upward position in which the rate of stored energy $\dot{H}$ is positive. Hence the trajectory never goes into that region. To make the region small i.e. to increase the amplitude of oscillation we take the reference energy E_{ref} a bit higher value than the actual value, by which the control input at each instant increases due to increase in difference $E - E_{ref}$.

4.4 Simulation and Experimental Results of Passivity Based Swing up

The Figures 4.5 and 4.6 show the simulation and experimental results of passivity based swing up control on inertia wheel pendulum. In simulation results shows the responses without and with pendulum friction. We set the reference energy, E_{ref} equal to the potential energy of the system at vertical upward position which is $E_{ref} = 2mgl$. The system parameters are given in the chapter 2. The gains k_e , k_v and k_u are chosen to be,

$$k_e = 4000, \quad k_v = 4, \quad k_u = 0.4$$

Figure 4.5 gives a comparison between simulation results and experimental results of pendulum angle, pendulum velocity and wheel angle, where as the Figure 4.6 gives a comparison between simulation and experimental results of wheel angular velocity, control input and pendulum energy. Now lets have a close observation of simulation and experimental results.

1. As the simulation result shows, the pendulum angle oscillates between $\pm\pi$ rad if we neglect pendulum friction, where as in the presence of friction it oscillates between ± 2.74 rad. The experimental result of pendulum angle exactly follows the response of simulation result with friction.
2. The maximum value of pendulum angular velocity for both simulation and experimental results are almost same.
3. As we observe from simulation result the wheel angle goes to zero as soon as the pendulum achieves its peak amplitude neglecting pendulum friction. Where as in presence of friction it oscillates between a some value. But the experimental results shows that the wheel angle goes on increasing, this is due to presence of a small positive average wheel angular velocity.
4. The wheel angle converges to zero during steady state if we neglect pendulum friction, where as it oscillates about zero with some amplitude in presence of friction. Experimental result also shows the small oscillations in wheel angular velocity.

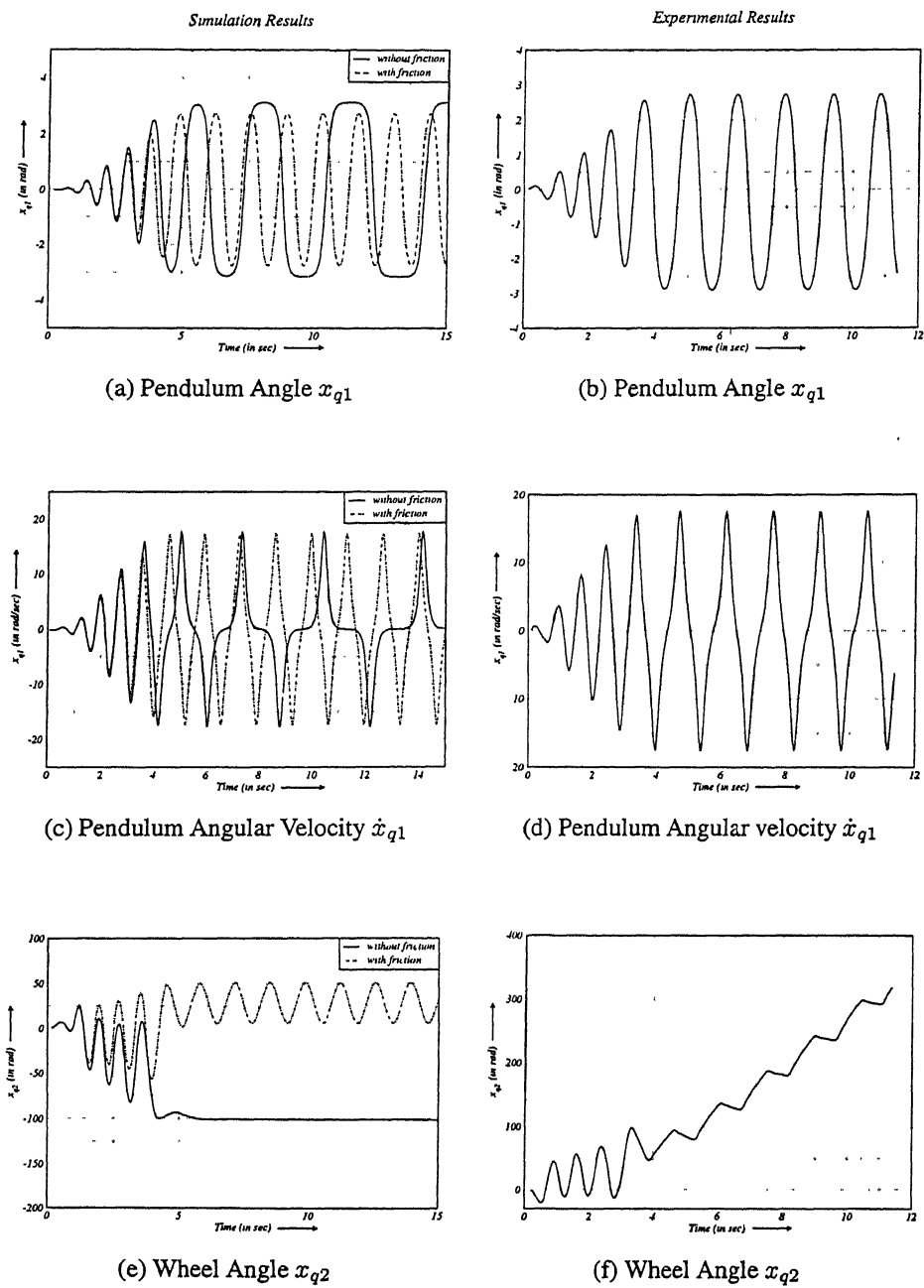
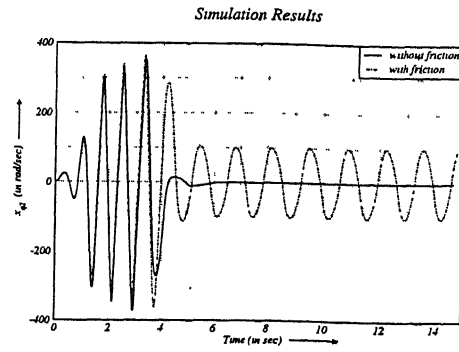
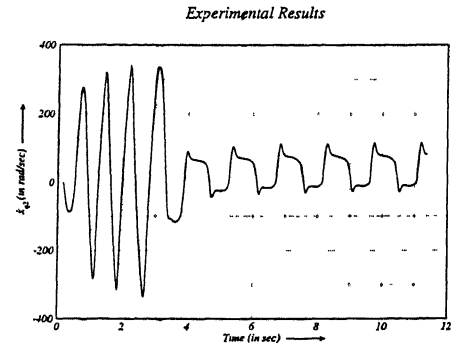


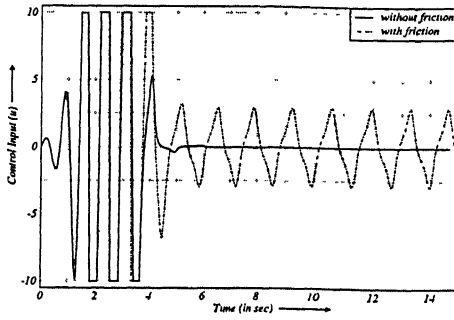
Figure 4.5: Simulation and Experimental Results of Passivity based swing up ($x_{q1} \sim t$, $\dot{x}_{q1} \sim t$, $x_{q2} \sim t$)



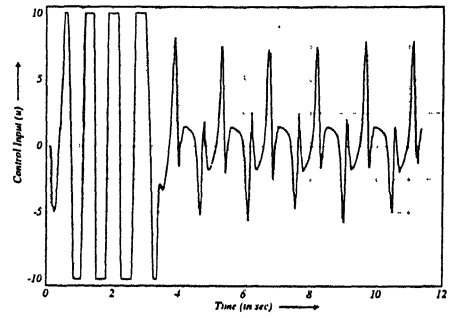
(a) Wheel Angular Velocity $\dot{x}_{q2}$



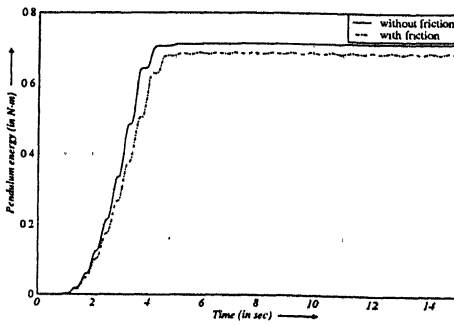
(b) Wheel Angular velocity $\dot{x}_{q2}$



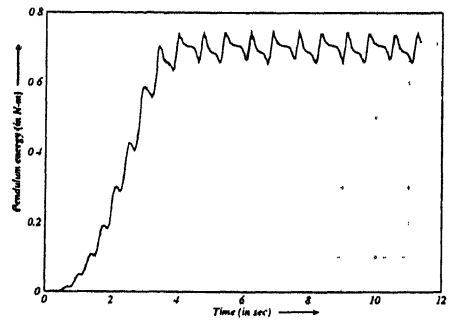
(c) Torque input u



(d) Torque Input u



(e) Pendulum Energy E



(f) Pendulum Energy E

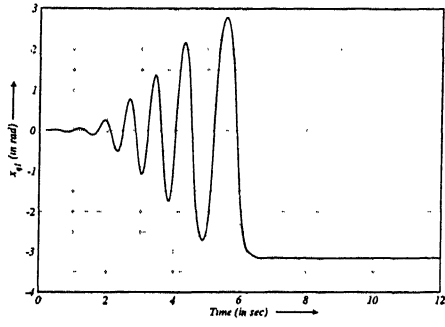
Figure 4.6: Simulation and Experimental results of Passivity Based swing up ($\dot{x}_{q2} \sim t$, $u \sim t$, $E \sim t$)

5. The control input ceases to zero as soon as pendulum oscillation achieves steady state value if we neglect pendulum friction, where as it oscillates about zero with some amplitude in presence of friction. It means energy is being supplied into the system to compensate the frictional losses with oscillation of pendulum.
6. In presence of friction there is a small steady state error in pendulum energy, where as the energy exactly converges to E_{ref} in case friction is neglected.

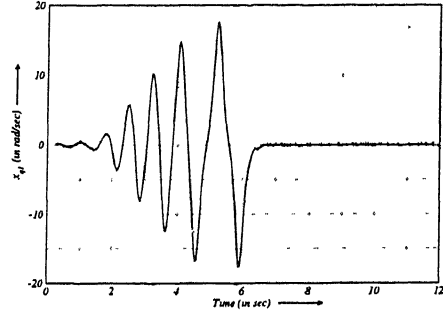
Figure (4.5(a)-4.5(e)) shows the simulation results without and with friction. The peak amplitude of swing up finally reaches to $\pm\pi$ and but with friction it goes up to $\pm 2.74rad$ (fig 4.5(a)) without friction. In absence of friction, wheel velocity goes to zero when the peak amplitude of swing up is reached, In presence of friction it oscillates with amplitude of $\pm 100rad/sec$ (fig 4.5(c)), To maintain this wheel velocity, torque also oscillates within a range (fig 4.5(d)). The stored energy never reaches $2mgl$ due to the presence of friction (fig 4.5(e)). Figure(4.5(f)) shows the phase plot of pendulum angle with pendulum angular velocity.

4.5 Experimental Results for Complete stabilisation

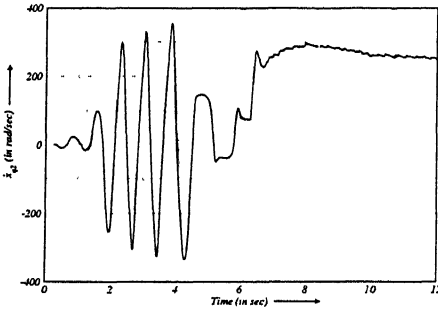
We are using a Linear State Feedback (LSF) controller to stabilize it at the vertical upward position. The responses with combined controller are shown in Figure 4.7. The balancing controller works within a band of ± 0.12 radian about the vertical upright position. So we need to bring the pendulum by swing up controller into a region of $x_{q1} \geq \pm 3.02159$ radian from which we switch over to Linear State Feedback (LSF) controller. But due to presence of friction the pendulum does not enter into the region. So we intentionally add a small amount of constant energy term to E_{ref} to make the region small in which $\dot{H}$ is positive. Hence the amplitude of pendulum oscillation increases to the desired value from which we can switch over to LSF control. Figure 4.7(f) shows the phase plot of pendulum angle with pendulum angular velocity with the combined swing up and balancing control



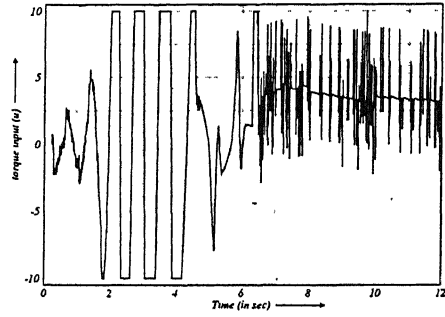
(a) Pendulum Angle x_{q1}



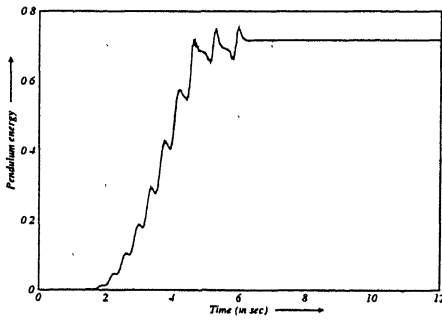
(b) Pendulum Angular Velocity $\dot{x}_{q1}$



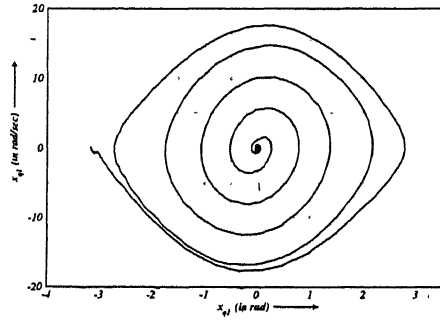
(c) Wheel Angular velocity $\dot{x}_{q2}$



(d) Torque input u



(e) Pendulum Energy



(f) Phase Plot $\dot{x}_{q1} \sim x_{q1}$

Figure 4.7: Experimental results of Passivity Based swing up with LSF balancing control($x_{q1} \sim t$, $\dot{x}_{q1} \sim t$, $x_{q2} \sim t$, $\dot{x}_{q2} \sim t$, $u \sim t$, $E \sim t$)

4.6 Concluding Remarks

In this chapter the application of non-linear control theory to the problem of swingup is investigated, the passivity based swing up control brings the pendulum from vertical downward position to a close region of vertical upward position. Experimental results shows that the swingup based on passivity responds quite faster compared to the Partial Feedback linearisation method described in chapter 2. Using passivity together with La Salle's invariance principle allowed us to analyse complete behavior of the closed loop system. With this method we can only swing up the pendulum to vertical upward position but to stabilize at the top we need a balancing controller.

* * *

Chapter 5

IDA-PBC Control of IWP

5.1 Introduction

The *Passivity-based Control* introduced by Ortega and Spong [33] defines a controller design methodology which achieves stabilization by passivation. More precisely, the control objective is to passivize the system with a storage function which has a minimum at desired equilibrium point. The method is very successful for the simple mechanical systems that can be stabilized shaping only the potential energy. In this case the closed loop system is still an EL system (as described in [34]) with the total energy being the difference between the stored and the supplied energies, hence the stabilization can be explained in terms of physical energy balancing. But in particular we are interested in system Inertia Wheel Pendulum, which requires shaping of total energy of the system for stabilization. The draw back of PBC approach for these type of systems is that the closed loop storage function (typically a quadratic function as chosen in equation 4.11 of chapter 4) are not the energy function in any meaningful physical sense. Although the closed loop system is passive but the closed loop system no longer remains an EL system.

In this chapter a new PBC theory called *Interconnection and damping assignment* (IDA) PBC [35] [29] which extends to a broader class of systems *e.g.* to applications that require shaping of the total energy. The special features of the method is that [36] (i) the physical structure (Hamiltonian) is preserved in closed loop, and (ii) the storage function of the passive map is precisely the total energy of the closed loop system. Some conditions (discussed later) on dissipation shows that total energy is the difference between the stored and the supplied

energies.

The main distinguishing feature between the *classical* PBC and IDA-PBC is that in former we select storage function to be assigned and then design the controller that ensures the objective (*i.e.* by rendering the energy function non-increasing). On the other hand, in IDA-PBC the closed loop energy function is obtained by the solution of partial differential equation (PDE) as a result of our choice of desired subsystems interconnections and damping. Though it is not easy to solve PDEs in general but, for inertia wheel pendulum the chosen interconnection and damping matrices with some explicit conditions makes it easy to get the solutions.

5.2 Port Controlled Hamiltonian systems

For the above we depart from the EL description of the system and consider *Port-controlled Hamiltonian* (PCH) models [37], which is a very large class of non-linear systems strictly containing the class of EL models. They result from the modeling of energy conserving lumped parameter physical systems with independent storage elements. Besides capturing the energy balance features of physical systems as in EL models, PCH models provides a classification of the variables (displacement and momentum) and the equations into those associated to phenomenological properties and those defining the interconnection structure related with the exchange of energy. They are therefore well suited to carry out the basic steps of PBC to shape energy function and adding dissipation.

5.2.1 Port-controlled Hamiltonian model from EL and Hamiltonian model

Let us recall the standard EL and Hamiltonian equation of motion. The standard EL equations are given as

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \right) - \frac{\partial L}{\partial q}(q, \dot{q}) = \tau \quad (5.1)$$

where $q = (q_1, \dots, q_k)^T$ are generalized configuration coordinates for the system with k degrees of freedom, the Lagrangian L equals the difference $K - P$ between kinetic energy K and potential energy P , and $\tau = (\tau_1, \dots, \tau_k)^T$ is the generalized forces acting on the system. Furthermore $\frac{\partial L}{\partial \dot{q}}$ and $\frac{\partial L}{\partial q}$ are the column vectors of partial derivatives of $L(q, \dot{q})$ with respect

to velocities $\dot{q}_1, \dots, \dot{q}_k$ and positions $q_1, \dots, q_k$ respectively. In standard mechanical systems the kinetic energy K is of the form

$$K(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} \quad (5.2)$$

where $k \times k$ inertia (generalized mass) matrix $M(q)$ is symmetric and positive definite for all q . In this case the vector of generalized *momenta* $p = (p_1, \dots, p_k)^T$, defined for any Lagrangian L as $p = \frac{\partial L}{\partial \dot{q}}$, is simply given by

$$p = M(q) \dot{q} \quad (5.3)$$

and by defining state vector $(q_1, \dots, q_k, p_1, \dots, p_k)^T$ (i.e the k second order equations of (5.1) transform into $2k$ first order equations as,

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p) \quad (= M^{-1}(q)p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p) + \tau \end{aligned} \quad (5.4)$$

where

$$H(q, p) = \frac{1}{2} p^T M^{-1}(q) p + P(q) \quad (= \frac{1}{2} \dot{q}^T M(q) \dot{q} + P(q)) \quad (5.5)$$

is the total energy of the system. The equations (5.4) are called the *Hamiltonian equations* of motion, and H is called *Hamiltonian*. The following energy balance immediately follows from (5.4):

$$\frac{d}{dt} H = \frac{\partial^T H}{\partial q}(q, p) \dot{q} + \frac{\partial^T H}{\partial p}(q, p) \dot{p} = \dot{q}^T \tau \quad (5.6)$$

which implies that the increase in energy of the system is equal to the supplied work (*conservation of energy*).

It should be noted that if H is considered as the storage function, then equation 5.6 implies that the system given by equation 5.4 is passive from input $u = \tau$ to output $y = \dot{q}$. However, the stored energy function H can be negative since it consists of the potential energy term $P(q)$ which can be also negative. The potential energy is always computed based on some reference level. This energy is usually positive above reference and negative below reference. If one can find the minimum potential energy of a system, then it is always possible to define

a positive term for the potential energy. Let the lower bound on the negative value of the potential energy is C . If we redefine the Hamiltonian as:

$$H(q, p) = \frac{1}{2} p^T M^{-1}(q) p + P(q) - C \quad (5.7)$$

then the Hamiltonian H can be treated as the stored energy function since its value is always positive.

The system dynamics given by equation 5.4 can be written in more general form as

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p), & (q, p) &= (q_1, \dots, q_k, p_1, \dots, p_k) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p) + B(q)u, & u &\in \mathbb{R}^m \\ y &= B^T(q) \frac{\partial H}{\partial p}(q, p) & (= B^T(q)\dot{q}), & y \in \mathbb{R}^m \end{aligned} \quad (5.8)$$

Here $B(q)$ is the input force matrix, with $B(q)u$ representing the generalized forces resulting from the control inputs $u \in \mathbb{R}^m$. The state space of (5.8) with local coordinates (q, p) is usually called the phase space. And when $m < k$, the system becomes an *under-actuated* system. Because of the form of output equations $y = B^T(q)\dot{q}$, the energy balance equation takes the following shape:

$$\frac{dH}{dt}(q(t), p(t)) = u^T(t)y(t) \quad (5.9)$$

If H is defined as given in equation 5.7, the Hamiltonian system given by equation 5.8 is a lossless state space system or a passive system.

A major generalization of the class of Hamiltonian systems is to consider systems which are described in local coordinates as

$$\begin{aligned} \dot{x} &= J(x) \frac{\partial H}{\partial x}(x) + g(x)u, & x &\in \chi, u \in \mathbb{R}^m \\ y &= g^T(x) \frac{\partial H}{\partial x}(x), & y &\in \mathbb{R}^m \end{aligned} \quad (5.10)$$

Here $J(x)$ is an $n \times n$ (we changed $2k \times 2k$ to $n \times n$ to make it convenient) matrix with entries

depending smoothly on x , which is assumed to be skew-symmetric, *i.e.*

$$J(x) = -J^T(x), \quad (5.11)$$

and $x = (x_1, \dots, x_n)$ are local coordinates for an n -dimensional state space manifold χ . Because of equation (5.11), one can easily recover energy balance equation $\frac{dH}{dt}(x(t)) = u^T(t)y(t)$, showing that the system dynamics given by equation (5.10) is also lossless if $H \geq 0$. The system dynamics in equation (5.10) with J satisfying equation (5.11) is called a *Port-controlled Hamiltonian (PCH) system* [37] with *structure matrix* $J(x)$ and *Hamiltonian* H .

5.2.2 Basic properties of Port-controlled Hamiltonian systems

Port-controlled Hamiltonian (PCH) systems represented by equations (5.10) and (5.11) not only can be seen as generalization of classical Hamiltonian equations of motion, but they naturally arise from modeling of physical systems without dissipative elements. As described in equation (5.10), a port-controlled Hamiltonian system is defined by a state space manifold χ with (J, g, H) . The pair $(J(x), g(x)), x \in \chi$, captures the *interconnection structure* of the system, with $g(x)$ modeling in particular the *ports* of the system. Independently from the interconnection structure, the function $H : \mathbb{R}^n \rightarrow \mathbb{R}$ defines the total energy of the system. PCH systems have following properties:

1. PCH systems are intrinsically *modular* *i.e.* parallel and feedback interconnection of a number of PCH systems again defines a PCH system, with overall interconnection structure determined by the interconnection structures of the composing individual PCH systems together with the interconnection between the systems.
2. Hamiltonian for overall system consisting of interconnected PCH systems is just the sum of individual Hamiltonians.
3. From the structure matrix $J(x)$ of a port-controlled Hamiltonian system one can directly extract useful information about the dynamical properties of the system. Since the structure matrix is directly related to the modeling of the system, this information usually has a direct physical interpretation.

Port-controlled Hamiltonian systems with dissipation

The system dynamics given by equation (5.10) can be made dissipative by adding a term $R(x)$ as follows:

$$\begin{aligned}\dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x) \frac{\partial H}{\partial x}(x)\end{aligned}\quad (5.12)$$

where $R(x)$ is a positive semi-definite symmetric matrix, depending smoothly on x . In this case the energy balancing property of equation (5.8) takes the form

$$\frac{dH}{dt}(x(t)) = u^T(t)y(t) - \frac{\partial^T H}{\partial x}(x(t))R(x(t))\frac{\partial H}{\partial x}(x(t)) \leq u^T(t)y(t) \quad (5.13)$$

Thus the system becomes passive if the Hamiltonian H is given as per equation (5.7). The system dynamics given by (5.12) is called a *port-controlled Hamiltonian system with dissipation* (PCHD) [37]. In this case, the system consists of two geometric structures - (i) the internal power conserving interconnection structure given by $J(x)$, and (ii) an additional dissipative structure given by $R(x)$.

5.3 Interconnection and Damping Assignment PBC

The PCHD system dynamics (5.13) forms a more general model of which a PCH system is a special case, i.e when dissipative structure matrix $R(x) = 0$.

The energy balance equation (5.13) can be written in a different form by integrating both sides of this equation:

$$\underbrace{\int_0^t u^T(s)y(s)ds}_{\text{supplied}} = \underbrace{H(x(t)) - H(x(0))}_{\text{stored}} + \underbrace{\int_0^t \left[\frac{\partial^T H}{\partial x}(x(s))R(x(s))\frac{\partial H}{\partial x}(x(s)) \right] ds}_{\text{dissipated}} \quad (5.14)$$

which holds for all $t \geq 0$. If the energy function is bounded from below PCH systems define a passive operator from $u \mapsto y$ with storage function $H(x)$. In this case equation (5.14) expresses the fact that a passive system can not store more energy than that is supplied to it from the outside, with difference being the dissipated energy. Following properties of PCHD systems can be easily verified if the system dynamics are given as per equation (5.14):

1. *One can only extract a finite amount of energy from a passive system:* Since the terms $H(x(t))$ and $\int_0^t \left[\frac{\partial^T H}{\partial x}(x(s)) R(x(s)) \frac{\partial H}{\partial x}(x(s)) \right] ds$ are always positive, it can be shown from (5.14)) that the following relation holds good:

$$-\int_0^t u^T(s)y(s)ds \leq H(x(0)) \quad (5.15)$$

which shows that one can only extract a finite amount of energy from a passive system.

2. *The uncontrolled PCHD system will eventually converge to the state corresponding to minimum energy if the potential energy function is non-negative:* When $u(t) = 0$ in equation (5.14), the relation $H(x(t)) \leq H(x(0))$ holds good. This implies that the energy of the uncontrolled system is non-increasing. In fact, this energy will eventually decrease in presence of dissipation until the system trajectories reach the point of minimum energy.
3. *The controlled PCHD with negative output feedback control converges to equilibrium point faster compared to the uncontrolled PCHD:* Let the negative output feedback control be $u = -k_v y$. Equation (5.12) thus becomes:

$$\dot{x} = [J(x) - (R(x) + g(x)k_v g^T(x))] \frac{\partial H}{\partial x}(x) \quad (5.16)$$

If $k_v = k_v^T > 0$, i.e. k_v is a symmetric positive definite matrix, then the value of dissipative term is increasing. Due to increase in dissipative term, the rate derivative of the energy function, $\frac{dH}{dt}$, will decrease faster compared to uncontrolled PCHD. Thus the system will converge to equilibrium point faster.

4. *The PCHD system with negative output feedback can exhibit equilibrium points of in-*

terest: It is clear that an open loop PCHD system has its own equilibrium points that may not be of interests to a control engineer. However, introduction of suitable negative feedback can take the closed loop system to any desired equilibrium point.

5.4 The Control Problem in the Context of Passivity

Given the PCHD system (5.12) and a desired fixed equilibrium point x_* , it is desired to find out a control action¹ $u = \beta(x) + v$ such that the closed loop dynamics is a PCHD system satisfying the new energy balancing equation,

$$H_d(x(t)) - H_d(x(0)) = \int_0^t [v^T(s)y'(s)]ds - d_d(t) \quad (5.17)$$

where $H_d(x)$ is the desired closed loop stored energy function, which has a strict (local) *minimum* at x_* ². y' (which may be equal to y) is the new passive output, and the dissipation term is represented by some function $d_d(t) \geq 0$, which is necessary to increase the convergence rate for the system to reach the equilibrium point x_* . In equation (5.17) the term $H_d(x)$ becomes zero or minimum when $v = 0$, and $d_d(t) = H_d(x(0))$. This implies that as the dissipative term increases along the system trajectory, at $x = x_*$, the closed loop stored energy function becomes zero due to the fact that $d_d(t) = H_d(x(0))$ at this desired equilibrium point.

One of our main concerns is to understand the nature of the stabilization mechanism, and in particular what is the rôle of dissipation. We will give conditions under which the energy function of the closed-loop PCHD system is the difference between the stored and the supplied energy i.e. $H_d(x(t)) = H(x(t)) - \int_0^t u(s)y(s)ds$. In this case we will say that stabilization is achieved via *energy-balancing*.

¹Only the case of *static state-feedback* is considered in this work

²That is, there exists an open neighborhood $\mathcal{R}$ of x_* such that $H_d(x) > H_d(x_*)$ for all $x \in \mathcal{R}$

5.5 IDA-PB Controller design

The objective of a IDA-PBC is to find a static state-feedback control $u = \beta(x)$ ($v = 0$ is taken) such that the closed-loop dynamics is a PCH system with dissipation of the form,

$$\dot{x} = [J_d(x) - R_d(x)] \frac{\partial H_d}{\partial x}(x), \quad (5.18)$$

where the new energy function $H_d(x)$ has a strict minimum at the desired equilibrium x_* , $J_d(x) = -J_d^T(x)$ (skew symmetricity) and $R_d(x) = R_d^T(x) \geq 0$ (symmetric positive definite). The following method gives the theoretical part, which allows to effectively construct the controller.

Design procedure

Proposition 5.1 [35, 29] *Given are functions $J(x)$, $R(x)$, $H(x)$, $g(x)$ and the desired equilibrium to be stabilized at x_* . Assume that one can find functions $\beta(x)$, $R_a(x)$, $J_a(x)$ such that $J_d(x) \triangleq J(x) + J_a(x) = -J_d^T(x)$, $R_d(x) \triangleq R(x) + R_a(x) = R_d^T(x) \geq 0$, and a vector function $K(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying*

$$[J_d(x) - R_d(x)]K(x) = -[J_a(x) - R_a(x)] \frac{\partial H}{\partial x}(x) + g(x)\beta(x) \quad (5.19)$$

and such that

(i) (Integrability) $K(x)$ is the gradient of scalar function. That is,

$$\frac{\partial K}{\partial x}(x) = \left[\frac{\partial K}{\partial x}(x) \right]^T \quad (5.20)$$

(ii) (Equilibrium assignment) $K(x)$, at x_* , verifies

$$K(x_*) = -\frac{\partial H}{\partial x}(x_*) \quad (5.21)$$

(iii) (Lyapunov stability) *The Jacobian of $K(x)$, at x_* , satisfies the bound*

$$\frac{\partial K}{\partial x}(x_*) > -\frac{\partial^2 H}{\partial x^2}(x_*) \quad (5.22)$$

Under these conditions, the system (equation (5.12)) in closed loop with $u = \beta(x) + v$ will be a PCH system with dissipation of the form (equation (5.18)), with new energy function

$$H_d(x) \triangleq H(x) + H_a(x) \quad (5.23)$$

and

$$\frac{\partial H_a}{\partial x}(x) = K(x). \quad (5.24)$$

Furthermore, if $v(t) = 0$, x_* will be a (locally) stable equilibrium of the closed-loop. It will be asymptotically stable if, in addition, the largest invariant set under the closed-loop dynamics contained in

$$\left\{ x \in \mathbb{R}^n \mid \left[\frac{\partial H_d}{\partial x}(X) \right] R_d(x) \frac{\partial H_d}{\partial x}(x) = 0 \right\} \quad (5.25)$$

equals $\{x_*\}$. An estimate of its domain of attraction is given by the largest bounded level set $\{x \in \mathbb{R}^n \mid H_d(x) \leq c\}$ for some positive constant c .

Proof:

For every given $\beta(x)$, $J_a(x)$, $R_a(x)$, (and on any contractible neighborhood of $\mathbb{R}^n$), the solution of equation (5.19) is a gradient of the form (5.24) if and only if the integrability condition given in equation (5.20) is satisfied. Using equation (5.23) it is easy to see that, in this case, the closed-loop is a PCH system of the form (5.18) and total energy can be given as (5.23). We will now prove that, under (5.21), (5.22), the stability of the equilibrium is ensured. Notice that the equilibrium assignment condition (5.21) ensures $H_d(x)$ has an extremum at x_* , while the Lyapunov stability condition (5.22) shows that it is actually an isolated minimum. On the other hand, from (5.17) (with $v = 0$ and a suitably defined $d_d(t)$) we have that, along the trajectories of the closed-loop, $H_d(x(t))$ is non-increasing, hence it qualifies as a Lyapunov function, and we can conclude that x_* is a stable equilibrium. Asymptotic stability follows

immediately invoking La Salles invariance principle [24] [3] *i.e.* every solution starting from a set $\{x | \dot{H}_d(x) \leq 0\}$ finally goes to the largest invariant set contained in the set $\{x | \dot{H}_d(x) = 0\}$. Due to skew-symmetric property of $J_d(x)$, $[\frac{\partial H_d}{\partial x}(x)]^T J_d(x) [\frac{\partial H_d}{\partial x}(x)] = 0$. Hence the condition (5.25). Finally, to ensure the solutions remain bounded, we give the estimate of the *domain of attraction* [2].

Discussion

1. As given in [38] for the system of the form (5.12) we can fix $J_a(x)$ and $R_a(x)$ and then look for the solution of the PDE

$$\begin{aligned} g^\perp(x) [J(x) + J_a(x) - (R(x) + R_a(x))] \frac{\partial H_a}{\partial x}(x) \\ = -g^\perp(x) [J_a(x) - R_a(x)] \frac{\partial H}{\partial x}(x) \end{aligned}$$

in terms of $H_a(x)$, where $g^\perp(x)$ is a left annihilator of $g(x)$, *i.e.* $g^\perp(x)g(x) = 0$. This is a linear PDE of the form $A(x)\frac{\partial H_a}{\partial x}(x) = b(x)$. Then the control can be directly calculated using the formula

$$\begin{aligned} \beta(x) = [g^T(x)g(x)]^{-1} g^T(x) \{ [J(x) + J_a(x) - (R(x) + R_a(x))] \frac{\partial H_a}{\partial x}(x) \\ + [J_a(x) - R_a(x)] \frac{\partial H}{\partial x}(x) \} \end{aligned}$$

2. For the general case, we can fix $J_a(x)$, $R_a(x)$, then solve the algebraic equation (5.19) for $K(x)$, which is trivial if the matrix $[J(x) + J_a(x) - [R(x) + R_a(x)]]$ is invertible. Finally the integrability condition (5.20) define a PDE directly for the control $\beta(x)$.
3. If there is no clear physical considerations for the choice of $J_a(x)$, $R_a(x)$, we can simply select them to simplify the solution of the PDE. This procedure is adopted for inertia wheel pendulum to stabilise it about vertical upward position.

5.5.1 Application to mechanical systems

In this section, it will be shown how IDA-PBC can be applied to mechanical systems. Let the mechanical system under consideration has a total energy:

$$H(x) = \frac{1}{2}x_p^T M^{-1}(x_q)x_p + U(x_q) \quad (5.26)$$

where the state space has been partitioned into position and momenta components as $x \triangleq [x_q^T, x_p^T]^T$. The inertial matrix $M(x_q)$ satisfies the property of symmetric positive definiteness, i.e. $M(x_q) = M^T(x_q) > 0$. The potential energy $U(x_q)$ is made non-negative as given in equation (5.7). The PCHD description (5.12 or 5.19) corresponds to constant interconnection and input matrices of the form

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad g(x_q) = \begin{bmatrix} 0 \\ g_p(x_q) \end{bmatrix} \quad (5.27)$$

and it is assumed that the system has no damping, i.e. $R(x) = 0$.

It is required to stabilize the equilibrium point of the form $x_* = [x_{q*}^T, 0]^T$. According to *proposition 5.1* [35] matrices $J_a(x)$ and $R_a(x)$ has been assumed as:

$$\begin{aligned} J_a(x) &= \begin{bmatrix} 0 & J_1(x_q) \\ -J_1^T(x_q) & J_2(x) \end{bmatrix} = -J_a^T(x) \\ R_a(x) &= \begin{bmatrix} 0 & 0 \\ 0 & R_2(x) \end{bmatrix} = R_a^T(x) \geq 0 \end{aligned} \quad (5.28)$$

with $J_1(x_q)$, $J_2(x)$, $R_2(x)$ free parameters to be defined later. One thing to be specified here, for the above equality conditions to be satisfied the free parameters should be chosen in such a way that $J_2(x) = -J_2^T(x)$ and $R_2(x) = R_2^T(x)$ should be satisfied. Under these conditions,

the PDE (5.19) reduces to

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} K_q(x) \\ K_p(x) \end{bmatrix} = - \begin{bmatrix} 0 & J_1(x_q) \\ -J_1^T(x_q) & J_2(x) - R_2(x) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x_q}(x) + K_q(x) \\ \frac{\partial H}{\partial x_p}(x) + K_p(x) \end{bmatrix} + \begin{bmatrix} 0 \\ g_p(x_q)\beta(x) \end{bmatrix} \quad (5.29)$$

where we have also partitioned the gradient of $H_a(x)$ (5.24) as

$$K(x) \triangleq \begin{bmatrix} K_q(x) \\ K_p(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial H_a}{\partial x_q}(x) \\ \frac{\partial H_a}{\partial x_p}(x) \end{bmatrix} \quad (5.30)$$

From equations (5.12) and (5.26), one can observe that if the interconnection matrix $J_d(x)$ for closed loop PCH system is not modified (i.e. $J_d(x) = J(x)$ with the additional interconnection matrix $J_a(x)$ assigned to 0), then the kinetic energy of the closed loop system can not be shaped, the reason behind this is as follows:

Assigning additional interconnection matrix $J_a(x) = 0$, it is clear from first $\frac{n}{2}$ equations of (5.29), that

$$K_p(x) = 0$$

From *Proposition 5.1* it is known that

$$K_p(x) = \frac{\partial H_a(x)}{\partial x_p}$$

Hhence $\frac{\partial H_a(x)}{\partial x_p} = 0$ is true if closed loop interconnection is not modified, which tells that the additional energy function $H_a(x)$ is not a function of x_p , it is a function of x_q only, i.e. $H_a(x) = H(x_q)$, hence there is only potential energy term in the additional enrgy function $H_a(x_q)$. So only potential energy of the closed loop system can be shaped, hence it is not possible to shape the kinetic energy if interconnection matrix is not modified.

For fully actuated case where $g_p(x_q) = I$, one can assign arbitrary potential energy and thus can add damping in all input channels. The under-actuation puts severe constraint on the method to get arbitrary potential energy.

As the additional energy term $H_a(x)$ plays a measure role in shaping both kinetic and potential energy, thus the free parameters of matrices $J_1(x_q)$, $J_2(x)$ and $R_2(x)$ shown in equation (5.28) are chosen in such a way that the PDE of equation (5.29) admits a solution of the form

$$H_a(x) = \frac{1}{2}x_p^T N(x_q)x_p + H'_a(x_q) \quad (5.31)$$

for some system matrix $N(x_q)$. To shape the x_p component of $H_d(x)$ we require $M^{-1}(x_q) + N(x_q)$ to be positive definite. On the other hand, the potential energy term of $H_a(x)$ i.e. $H'_a(x_q)$ will need to dominate the open loop system potential energy term $U(x_q)$ to keep $H_d(x)$ positive definite. From the first $\frac{n}{2}$ equations of (5.29) an expression of matrix $N(x_q)$ in terms of $M(x_q)$ and $J_1(x_q)$ can be obtained. To solve last $\frac{n}{2}$ equations of (5.29) it is required to eliminate dependence with respect to x_p to find out the solution of PDEs. Here is the method to present our main result.

How to find out the function $H_a(x_q)$

Proposition 5.2 [29] *To find out $H_a(x_q)$ consider the mechanical system of the form (5.12) with the system total energy given in equation (5.26), the system matrices given in equation (5.27) and the desired equilibrium to be stabilized $x_* = [x_{q*}^T, 0]^T$. It is assumed that one can find out functions $\beta(x)$, $R_2(x) = R_2^T(x) \geq 0$, $J_1(x)$, $J_2(x) = -J_2^T(x)$ and a vector function $K'(x_q) : \mathbb{R}^{\frac{n}{2}} \rightarrow \mathbb{R}^{\frac{n}{2}}$ satisfying,*

$$\begin{aligned} [I + J_1^T(x_q)]K'_q(x_q) &= -J_1^T(x_q)\frac{\partial H}{\partial x_q}(x) + \frac{1}{2}[I + J_1^T(x_q)]\frac{\partial}{\partial x_q}\{S(x)\} \\ &\quad + [J_2(x) - R_2(x)][I + J_1^T(x_q)]^{-1}M^{-1}(x_q)x_p \\ &\quad - g_p(x_q)\beta(x) \end{aligned} \quad (5.32)$$

(where $S(x) = x_p^T[I + J_1^T(x_q)]^{-1}J_1(x_q)M^{-1}(x_q)x_p$)

and such that

(i) (Integrability) $K'_q(x_q)$ is the gradient of a scalar function.

(ii) (Equilibrium assignment) $K'_q(x_q)$ at x_q verifies

$$K'_q(x_{q*}) = -\frac{\partial^2 U}{\partial x_q^2}(x_{q*}) \quad (5.33)$$

(iii) (Lyapunov stability) The Jacobian of $K'_q(x_{q*})$, at x_{q*} , satisfies the bound

$$\frac{\partial K'_q}{\partial x_q}(x_q) > -\frac{\partial^2 U}{\partial x_q^2}(x_{q*}) \quad (5.34)$$

and $[I + J_1^T(x_q)]^{-1}M^{-1}(x_q)$ is a symmetric positive definite matrix.

Under these conditions the system (5.12) in closed loop with $u = \beta(x)$ is a PCH system with total energy,

$$H_d(x) = \frac{1}{2}x_p^T[I + J_1(x_q)]^{-1}M^{-1}(x_q)x_p + U(x_q) + H'_a(x_q) \quad (5.35)$$

where

$$\frac{\partial H'_a}{\partial x_q}(x_q) = K'_q(x_q) \quad (5.36)$$

and the interconnection and damping matrices are of the form

$$J_d(x) = \begin{bmatrix} 0 & I + J_1(x_q) \\ -I - J_1^T(x_q) & J_2(x) \end{bmatrix} = -J_d^T(x)$$

$$R_d(x) = \begin{bmatrix} 0 & 0 \\ 0 & R_2(x) \end{bmatrix} = R_d^T \geq 0$$

respectively. Furthermore, x_* will be a locally stable equilibrium. It will be asymptotically stable if condition of equation (5.25) satisfies i.e in addition to above the largest invariant set under the closed loop dynamics contained in

$$\left\{ x \in \mathbb{R}^n \mid \left[\frac{\partial H_d}{\partial x}(x) \right]^T R_d(x) \frac{\partial H_d}{\partial x}(x) = 0 \right\}$$

equals $\{x_*\}$.

Proof:

To carry out the proof it is required to show that, for this particular case of the mechanical system, the conditions of general proposition 5.1 reduces to those of proposition 5.2

From the first $\frac{n}{2}$ equations of (5.29) it is obtained that

$$\begin{aligned} K_p(x) &= -J_1(x_q) \left[\frac{\partial H}{\partial x_p}(x) + K_p(x) \right] \\ &= -J_1(x_q) [M^{-1}(x_q)x_p + K_p(x)] \end{aligned} \quad (5.37)$$

As $K_p(x) = \frac{\partial H}{\partial x_p}$, postulating a solution of $H_a(x)$ of the form (5.31) one can get that $K_p(x) = N(x_q)x_p$, so equation (5.37) becomes $N(x_q) = -J_1(x_q)[M^{-1}(x_q) + N(x_q)]$ or $N(x_q) = -[I + J_1(x_q)]^{-1}J_1M^{-1}(x_q)$. Assuming invertibility of $[I + J_1(x_q)]$, one can see, this is equivalent to

$$\begin{aligned} M^{-1}(x_q) + N(x_q) &= J_1^{-1}(x_q)N(x_q) \\ \Rightarrow M^{-1}(x_q) + N(x_q) &= [I + J_1(x_q)]^{-1}M^{-1}(x_q) \end{aligned} \quad (5.38)$$

which, under conditions of proposition is symmetric and positive definite. Hence, in its x_p coordinates $H_d(x)$ (5.35) has a minimum at the desired point.

For the last $\frac{n}{2}$ equations of (5.29), with the postulated solution of (5.31) it can be determined that

$$K_q(x) = \frac{1}{2} \frac{\partial}{\partial x_q} \{x_p^T N(x_q)x_p\} + K'_q(x_q) \quad (5.39)$$

where $K'_q(x_q)$ is defined by (5.36).

As

$$\begin{aligned} K_q(x) &= \frac{\partial H_a}{\partial x_q} = \frac{\partial}{\partial x_q} \left[\frac{1}{2} x_p^T N(x_q)x_p + H'_a(x_q) \right] \\ &= \frac{1}{2} \frac{\partial}{\partial x_q} [x_p^T N(x_p)x_p] + K'_q(x_q) \quad \left(\because \frac{\partial H'_a}{\partial x_q} = K'_q(x_q) \right) \end{aligned} \quad (5.40)$$

And from last $\frac{n}{2}$ equations of (5.29) it is clear that

$$\begin{aligned} -K_q(x) &= J_1^T(x_q) \left[\frac{\partial H}{\partial x_q} + K_q(x) \right] \\ &\quad - (J_2(x) - R_2(x)) \left[\frac{\partial H}{\partial x_p} + K_p(x) \right] + g_p(x_q) \beta(x) \end{aligned} \quad (5.41)$$

So replacing $K_q(x)$ by (5.40) and replacing $N(x_q)$ by $-J_1(x_q)[M^{-1}(x_q) + N(x_q)]$ one can get final expression of (5.32). The proof is completed noting that under the remaining conditions of the proposition 5.2 the function $H_d(x)$ (5.35) has a local isolated minimum at x_* .

5.5.2 Inertia Wheel Pendulum System

In this section the Proposition 5.1 is applied to Inertial Wheel Pendulum to stabilize it at vertical upward position. It is shown here that IDA-PBC defines a continuous control law that “almost” globally stabilizes the upward position, if no actuator saturation takes place. The “almost” term makes the controller conservative because the basin of attraction of the desired equilibrium point is an open dense set in the state space, which excludes some initial conditions with which the state trajectories do not converge to the desired equilibrium point.

Model

The dynamic equations of inertia wheel pendulum with the angle (pendulum angle and wheel angle) reference along vertical upward axis can be written as (from equation (2.8) of Chapter 2),

$$\begin{bmatrix} j & 0 \\ 0 & j_r \end{bmatrix} \begin{bmatrix} \ddot{x}_{q1} \\ \ddot{x}_{q2} \end{bmatrix} + \begin{bmatrix} -mgl \sin x_{q1} \\ 0 \end{bmatrix} = \begin{bmatrix} -\tau \\ \tau \end{bmatrix} \quad (5.42)$$

where $[x_{q1}, x_{q2}]^T = [\psi_p, \psi_p + \psi_r]^T$ of which ψ_p, ψ_r are pendulum and rotor angle respectively. At vertical upward position $[\psi_p, \psi_r] = [0, 0]$. Defining the state $x = [x_q^T, x_p^T]^T = [x_q^T, M\dot{x}_q^T]^T$, with $M = \text{diag}\{j, j_r\}$, the total energy is

$$H(x) = \frac{1}{2} x_p^T M^{-1} x_p + mgl(\cos x_{q1} - 1) \quad (5.43)$$

and the PCH model takes the form of (5.12) with $g_p(x_q) = [-1, 1]^T$, which can be shown as,

$$\begin{bmatrix} \dot{x}_{q1} \\ \dot{x}_{q2} \\ \dot{x}_{p1} \\ \dot{x}_{p2} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} -mgl \sin x_{q1} \\ 0 \\ \frac{x_{p1}}{J} \\ \frac{x_{p2}}{J_r} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \tau \quad (5.44)$$

The equilibrium to be stabilized is the upward position with the inertia disk aligned, which corresponds to $x_* = 0$.

Controller Design

The IDA-PBC is designed here in two steps. First a state feedback control is determined which shapes the closed loop energy to stabilize the upward position of pendulum, finally the damping is added to closed loop system to ensure asymptotic stability. The next task is to find out the region in which the controller works for the constrained control input Inertial Wheel Pendulum system.

Energy Shaping

For the inertia wheel pendulum system, the inertia matrix M is independent of x_q . Hence recalling the discussion in the proposition 5.1, one can take $J_2(x) = R_2(x) = 0$. Also it will be clear very soon that, taking J_1 to be a constant matrix finding the solution of PDEs become easier. With these considerations the PDE to be solved (5.29) takes the form

$$\begin{bmatrix} 0 & I + J_1 \\ -(I + J_1)^T & 0 \end{bmatrix} \begin{bmatrix} K_q \\ K_p \end{bmatrix} = \begin{bmatrix} -J_1 M^{-1} x_p \\ J_1^T \begin{bmatrix} -mgl \sin x_{q1} \\ 0 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0 \\ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{bmatrix} \tau \quad (5.45)$$

Separating out the equations one can get, for first two rows $J_1(M^{-1}X_p + K_p) = -K_p$. A solution of the form $H_a = \frac{1}{2}x_p^T N x_p + H'_a(x_q)$ is postulated for the equation (5.45) according to equation (5.31), with $N = N^T$, and $H'_a(x_q)$ to be defined later. In this case one can easily get $J_1(M^{-1} + N)x_p = -N x_p$ which, under the assumption of invertibility of $(I + J_1)$, is

satisfied if and only if $M^{-1} + N = (I + J_1)^{-1}M^{-1}$. Let

$$J_1 = \begin{bmatrix} a & d \\ b & 0 \end{bmatrix}$$

with a, b, d constants to be assigned by defining some conditions. As $N = -(I + J_1)^{-1}J_1M^{-1}$, substituting matrix parameters of J_1 and M one can easily get

$$N = \begin{bmatrix} \frac{bd-a}{j(1+a-bd)} & \frac{-d}{j_r(1+a-bd)} \\ \frac{-b}{j(1+a-bd)} & \frac{bd}{j_r(1+a-bd)} \end{bmatrix}$$

symmetric condition $N = N^T$ imposes $b = \frac{j}{j_r}d$, so matrix N becomes

$$N = \begin{bmatrix} \frac{\frac{j}{j_r}d^2-a}{j(1+a-\frac{j}{j_r}d^2)} & \frac{-d}{j_r(1+a-\frac{j}{j_r}d^2)} \\ \frac{-d}{j_r(1+a-\frac{j}{j_r}d^2)} & \frac{j d^2}{j_r^2(1+a-\frac{j}{j_r}d^2)} \end{bmatrix}$$

So sum $M^{-1} + N$ comes out to be

$$M^{-1} + N = \begin{bmatrix} \frac{1}{j(1+a-\frac{j}{j_r}d^2)} & \frac{-d}{j_r(1+a-\frac{j}{j_r}d^2)} \\ \frac{-d}{j_r(1+a-\frac{j}{j_r}d^2)} & \frac{1+a}{j_r(1+a-\frac{j}{j_r}d^2)} \end{bmatrix}$$

which needs to be positive definite to shape the x_p components of $H_d(x)$. To satisfy this condition following inequality must be true

$$1 + a - \frac{j}{j_r}d^2 > 0 \quad \dots \text{Condition I} \quad (5.46)$$

Now looking into third and fourth equations of (5.45) which is in form of

$$-(I + J_1)^T K_q = J_1^T \begin{bmatrix} -mgl \sin x_{q1} \\ 0 \end{bmatrix} + \begin{bmatrix} -\tau \\ \tau \end{bmatrix} \quad (5.47)$$

and replacing J_1, b and adding up both the rows, a simple PDE in following form is obtained

$$(1 + a + d)K_{q1} + (1 + \frac{j}{j_r}d)K_{q2} = (a + d)mgl \sin x_{q1}$$

If $(1 + a + d) \neq 0$ and $(1 + \frac{j}{j_r}d) \neq 0$, then this PDE admits a family of solutions

$$H'_a(x_q) = -\frac{a+d}{1+a+d}mgl(\cos x_{q1} - 1) + \Phi(z) \quad (5.48)$$

where $\Phi(\cdot)$ is an arbitrary (continuously differentiable) function with argument

$$z \triangleq x_{q1} - \frac{1+a+d}{1+\frac{j}{j_r}d}x_{q2}$$

So the new potential energy (Let's say it $U_N(x)$)

$$U_N(x) = mgl(\cos x_{q1} - 1) + H'_a = \frac{mgl}{1+a+d} \cos x_{q1} - mgl + \Phi(z)$$

This new potential energy function will have an *extremum* at $x_q = 0$ if and only if

$$\begin{aligned} \left[\frac{\partial U_N}{\partial x_{q1}}(x) \right]_{x_q=0} &= \left[-\frac{mgl}{1+a+d} \sin x_{q1} + \frac{\partial \Phi(z)}{\partial z} \right]_{x_q=0} = 0 \\ &= \left[\frac{\partial \Phi(z)}{\partial z} \right]_{x_q=0} = 0 \end{aligned} \quad (5.49)$$

and

$$\left[\frac{\partial U_N}{\partial x_{q2}}(x) \right]_{x_q=0} = -\frac{1+a+d}{1+\frac{j}{j_r}d} \left[\frac{\partial \Phi(z)}{\partial z} \right]_{x_q=0} = 0 \quad (5.50)$$

Hence from the equations (5.49) and (5.50) the following condition is true

$$\frac{\partial \Phi(0)}{\partial z} = 0 \quad (5.51)$$

It will, furthermore, be a minimum if $\frac{\partial^2 U_N}{\partial x_q^2}$ is positive i.e., the matrix

$$\begin{bmatrix} \frac{\partial^2 U_N}{\partial x_{q1}^2} & \frac{\partial^2 U_N}{\partial x_{q1} \partial x_{q2}} \\ \frac{\partial^2 U_N}{\partial x_{q2} \partial x_{q1}} & \frac{\partial^2 U_N}{\partial x_{q2}^2} \end{bmatrix}$$

is positive definite. Hence following conditions must be satisfied

$$\begin{aligned} \left[\frac{\partial^2 U_N}{\partial x_{q1}^2} \right]_{x_q=0} &= \left[-\frac{mgl}{1+a+d} \cos x_{q1} + \frac{\partial^2 \Phi(z)}{\partial z^2} \right]_{x_q=0} > 0 \\ \Rightarrow \left[\frac{\partial^2 \Phi(z)}{\partial z^2} \right]_{x_q=0} &> \frac{mgl}{1+a+d} \end{aligned} \quad (5.52)$$

and

$$-(1 + a + d) \frac{\partial^2 \Phi(z)}{\partial z^2} > 0$$

Clearly, a necessary condition for this is

$$1 + a + d < 0 \quad \dots \text{Condition II} \quad (5.53)$$

In which case it is sufficient to take

$$\frac{\partial^2 \Phi(0)}{\partial z^2} > 0 \quad (5.54)$$

To verify (5.52) and (5.54) a quadratic function $\Phi(z) = \frac{K'_p}{2} z^2$ has been proposed with $K'_p > 0$. Finally, to verify the conditions of proposition 5.1 the parameters a and d should satisfy conditions given in equations (5.46) and (5.53) whose intersection is clearly a non empty set. So the final expression of additional energy term H_a is,

$$H_a = \frac{1}{2} x_p^T N x_p - \frac{a + d}{1 + a + d} mgl (\cos x_{q1} - 1) + \frac{K'_p}{2} (x_{q1} + \gamma_2 x_{q2}) \quad (5.55)$$

The energy shaping stage of the design is completed evaluating the control law from (5.47) (*Taking any one of the two equations*), taking the first equation of (5.47), i.e.

$$-(1 + a)K_{q1} - bK_{q2} = -amgl \sin x_{q1} - \tau$$

and substituting functions K_{q1} , K_{q2} , b , $\Phi(z)$ and z one can easily get the energy shaping control law as follows:

$$\tau = \gamma_1 \sin x_{q1} + K_p (x_{q1} + \gamma_2 x_{q2}) \quad (5.56)$$

where $K_p \triangleq \frac{1+a-\frac{1}{jr}d^2}{1+\frac{1}{jr}d} K'_p > 0$ and

$$\gamma_1 \triangleq \frac{dmgl}{1+a+d}, \quad \gamma_2 \triangleq -\frac{1+a+d}{1+\frac{1}{jr}d}$$

Damping Injection

The controller above yields the *conservative* closed loop system $\dot{x} = J_d \frac{\partial H_d}{\partial x}(x)$, $J_d = -J_d^T$, with the origin a stable equilibrium, because $\{0\} = \arg \min\{H_d(x)\}$. The equilibrium point can be made *asymptotically* stable with damping injection, which is achieved by feeding back

the passive output $y' = g^T(x) \frac{\partial H_d}{\partial x}(x)$, the passive output y' of closed loop system is determined as follows:

$$y' = \begin{bmatrix} 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H_d}{\partial x_{q1}} \\ \frac{\partial H_d}{\partial x_{q2}} \\ \frac{\partial H_d}{\partial x_{p1}} \\ \frac{\partial H_d}{\partial x_{p2}} \end{bmatrix}$$

Which comes out to be

$$y' = -\frac{1 + \frac{j}{j_r}d}{j \left(1 + a - \frac{j}{j_r}d^2\right)} \left(x_{p1} + \gamma_2 \frac{j}{j_r}x_{p2}\right) \quad (5.57)$$

Taking $-\frac{1 + \frac{j}{j_r}d}{j \left(1 + a - \frac{j}{j_r}d^2\right)} = \kappa$, the complete control law is

$$\tau = \gamma_1 \sin x_{q1} + K_p(x_{q1} + \gamma_2 x_{q2}) - k_v y' \quad (5.58)$$

where k_v is a positive proportionality constant, the constant κ is always negative due to the conditions 5.46 and 5.53. So the above equation (5.58) can be rewritten as,

$$\tau = \gamma_1 \sin x_{q1} + K_p(x_{q1} + \gamma_2 x_{q2}) + K_v \left(x_{p1} + \gamma_2 \frac{j}{j_r}x_{p2}\right) \quad (5.59)$$

where $K_p, K_v > 0$ and $K_v = -\kappa k_v$. Then control input to the PWM card is $u = \tau/0.00493$ [as discussed in Chapter 2]. By substituting τ from equation (5.59) in place of $\beta(x)$ in equation (5.19) and replacing matrices J, J_a by their elements, we can represent the additional negative output feedback term in control input τ in terms of elements of damping matrix R_a .

Asymptotic Stability

It is required to prove that the vertical upward position of pendulum is asymptotically stable which is proved in terms of convergence of closed loop energy function H_d . The closed loop energy for mechanical systems is shown in equation (5.35). So closed loop energy for the

pendulum system is calculated as

$$H_d = \frac{1}{2} x_p^T (M^{-1} + N) x_p + \frac{\gamma_1}{d} (\cos x_{q1} - 1) + \frac{K'_p}{2} (x_{q1} + \gamma_2 x_{q2})^2 \quad (5.60)$$

As $[M^{-1} + N]$ is positive definite, $d < 0$ and $-1 \leq (\cos x_{q1}) \leq 1$, thus all terms of H_d are ≥ 0 , which states that $H_d \geq 0$. Time derivative of closed loop energy function H_d is,

$$\dot{H}_d = \frac{\partial H_d}{\partial x_{q1}} \dot{x}_{q1} + \frac{\partial H_d}{\partial x_{q2}} \dot{x}_{q2} + \frac{\partial H_d}{\partial x_{p1}} \dot{x}_{p1} + \frac{\partial H_d}{\partial x_{p2}} \dot{x}_{p2}$$

Substituting the partial derivative terms and time derivative of states in the above equation and simplifying one can get,

$$\dot{H}_d = -\frac{K'_p K_v}{j K_p} (x_{p1} + \gamma_2 \frac{j}{j_r} x_{p2})^2 \quad (5.61)$$

which is a negative semidefinite function, this implies that H_d is a non-increasing function. Hence H_d finally converges to a constant. As energy H_d is remaining bounded, observing equation (5.60) one can easily get that all states of the system *i.e.* $(x_{q1}, x_{q2}, x_{p1} \text{ and } x_{p2})$ will remain bounded. Thus LaSalle's invariance principle can be applied here[24]. In order to apply LaSalle's theorem, it is required to define a compact set Ω with the property that every solution starts in this set remains in it for all future time. Again it is required to define an invariant set Γ , the set of all the points at which $\dot{H}_d = 0$. From equation(5.61) in set Γ , $\dot{H}_d = 0$. And from this, it follows that $(x_{p1} + \gamma_2 \frac{j}{j_r} x_{p2}) = 0$ in Γ . Which implies, in the invariant set Γ

$$x_{p2} = -\frac{j_r}{j \gamma_2} x_{p1} \quad (5.62)$$

So in invariant set the control input τ takes the form

$$\tau = \gamma_1 \sin x_{q1} + K_p (x_{q1} + \gamma_2 x_{q2}) \quad (5.63)$$

As in invariant set $\dot{H}_d = 0$, it means H_d is constant in invariant set, by carefully observing the closed loop energy function H_d in equation (5.60) it is clear that H_d will remain constant when all the terms involved in it remain constant. From this observation it is clear that x_{q1} is a constant, thus $x_{p1} = 0$ and hence equation (5.62) shows that x_{p2} is also 0.

The rate of system energy can be determined from equation (5.43) to be $\dot{H} = \left(-\frac{x_{p1}}{j} + \frac{x_{p2}}{j_r}\right) \tau$, as both x_{p1} and x_{p2} are equal to 0 in invariant set Γ , thus $\dot{H} = 0$ in the set. This implies H must be a constant. Two cases for the system energy are considered in invariant set, those are $H = 0$ and $H = K \neq 0$ for some constant K .

1. $H = 0$: As $\dot{H} = 0$, from equation(5.43) we get As $x_{p1} = 0$, hence $x_{p2} = 0$. So kinetic energy of the system $\frac{1}{2}x_p^T M^{-1}x_p = 0$. The system energy

$$H = \frac{1}{2}x_p^T M^{-1}x_p + mgl(\cos x_{q1} - 1) = 0 \quad (5.64)$$

Making use of equation (5.62) we get

$$H = \frac{1}{2} \left(1 + \frac{j_r}{j\gamma_2^2}\right) \dot{x}_1^2 + mgl(\cos x_{q1} - 1) = 0$$

Since $\frac{1}{2} \left(1 + \frac{j_r}{j\gamma_2^2}\right)$ is constant, above equation defines a particular trajectory called Homoclinic orbit of the pendulum in the $(x_{q1}, \dot{x}_{q1})$ phase plane. Hence pendulum moves in clockwise or anti-clockwise to reach equilibrium point. Observing equation (5.64) we get the potential energy of the system $mgl(\cos x_{q1} - 1) = 0$. Hence $x_{q1} = k\pi$ for k even. So equation(5.60) takes the form $H_d = \frac{K'_p}{2}(x_{q1} + \gamma_2 x_{q2})^2$ which must be a constant, to which the closed loop energy finally converges (as $\dot{H}_d$ is non-increasing). This is the potential energy of the closed loop system when $mgl(\cos x_{q1} - 1) = 0$. As per the design (in equation(5.51)) of the controller the closed loop potential energy U_N is minimum at $z = (x_{q1} + \gamma_2 x_{q2}) = 0 \Rightarrow x_{q2} = -\frac{k\pi}{\gamma_2}$. Hence equation (5.59) shows τ is also = 0. So the position of pendulum with states $[k\pi, -\frac{k\pi}{\gamma_2}, 0, 0]$ for k even, which is the upward position of the pendulum, is asymptotically stable as this position is the only largest invariant set in set Γ

2. $H = K \neq 0$: Since in invariant set $x_{p1} = 0$, kinetic energy of the system $\frac{1}{2}x_p^T M^{-1}x_p = 0$, this implies that potential energy $mgl(\cos x_{q1} - 1)$ is equal to constant K , which gives x_{q1} is constant. As the kinetic energy of closed loop system is 0 (as $x_{p1} = 0$), closed loop potential energy $U_N = \frac{\gamma_1}{d}(\cos x_{q1} - 1) + \frac{K'_p}{2}(x_{q1} + \gamma_2 x_{q2})^2$ which must be a constant and minimum, to which closed loop energy H_d converges. As per the controller design

U_N is minimum at $z = (x_{q1} + \gamma_2 x_{q2}) = 0$. With $x_{p1} = 0$, from the system dynamic equation (5.45) one can get,

$$\begin{aligned}\dot{x}_{p1} &= mgl \sin x_{q1} - u \\ &= mgl \sin x_{q1} - \gamma_1 \sin x_{q1} + K_p(x_{q1} + \gamma_2 x_{q2}) \quad \text{as } x_{p1} = 0 \\ &= \frac{1+a}{1+a+d} mgl \sin x_{q1} - K_p(x_{q1} + \gamma_2 x_{q2})\end{aligned}\quad (5.65)$$

As x_{q1} is constant, $\dot{x}_{p1} = 0$. Substituting $z = (x_{q1} + \gamma_2 x_{q2}) = 0$ in equation (5.65) one can get $\frac{1+a}{1+a+d} mgl \sin x_{q1} = 0$, which gives $x_{q1} = k\pi$ for k even $\Rightarrow x_{q2} = -\frac{k\pi}{\gamma_2}$. Hence the pendulum system energy $H = K = 0$, which gives the same conclusion as first case.

5.5.3 Simulation Results for IDA-PBC on IWP

Response of the pendulum with IDA-PBC are shown in figures [5.1-5.2]. The system parameters remain same as specified in Chapter 1. The controller is tuned to satisfy the Conditions I (5.46) and II (5.53) *i.e.*

$$1 + a - \frac{j}{j_r} d^2 > 0 \text{ and } (1 + a + d) < 0$$

Taking the specified parameters $j = 4.572 \times 10^{-3}$ and $j_r = 2.495 \times 10^{-5}$ the above two conditions can be combined as

$$-\frac{j}{j_r} d^2 = 183.2465 d^2 < 1 + a < -d \quad (5.66)$$

The parameters a , d and c should be chosen to satisfy above condition (5.66) and also to make the controller response faster than the system response. Or in physical sense, it can be said that around vertical upward position the controller must be faster enough to pull the pendulum to exact upward position, before it falls due to any small external disturbances like noise. Keeping these two things in mind, a and d are tuned to -0.9979 and -0.003 respectively. With these parameters we get $\gamma_1 = 1.194816$, $\gamma_2 = 0.001999$ and $\kappa = -218469.846066$. Choosing $K'_p = 0.1 > \frac{mgl}{1+a+d}$ and $k_v = 0.0001$ we get $K_p = 0.0001$ and $K_v = 21.846985$.

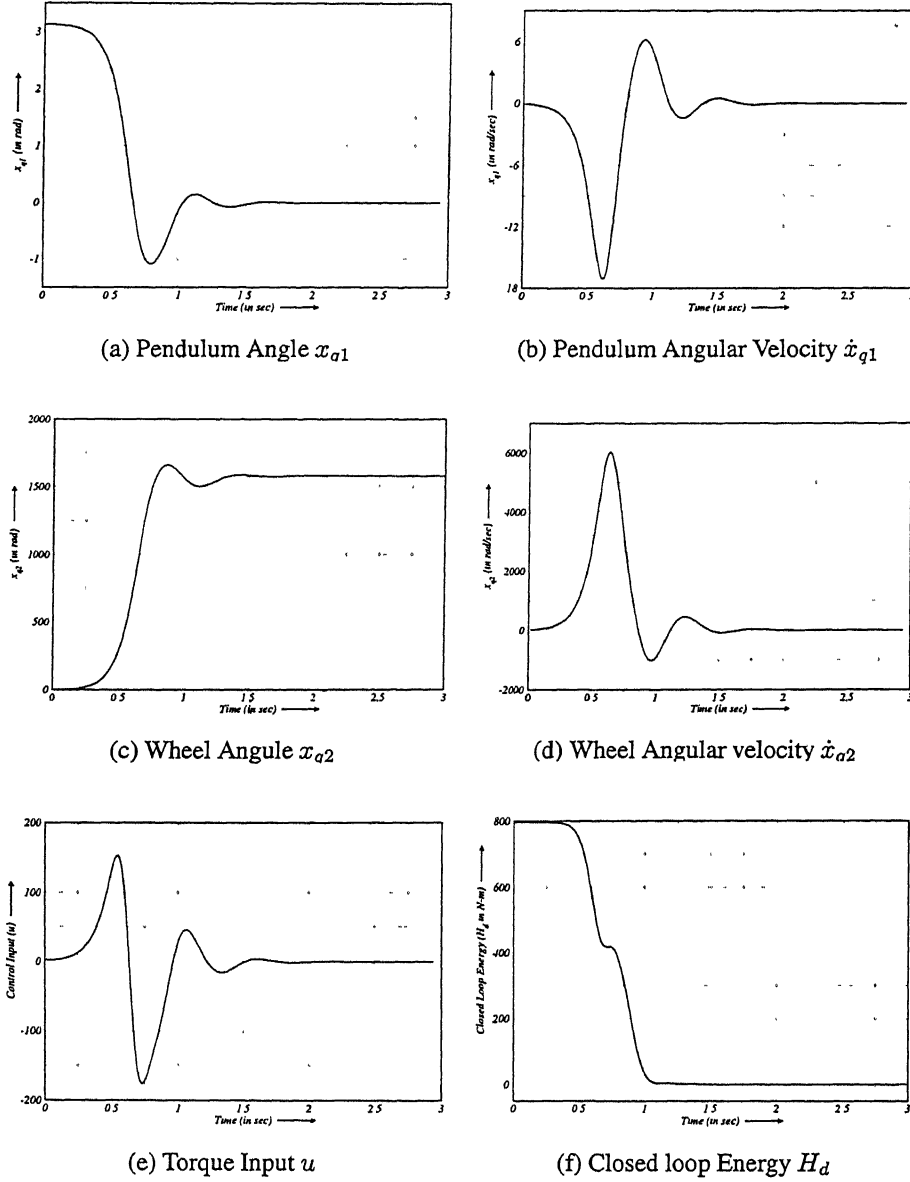


Figure 5.1: Simulation results of IDA-PBC ($x_{q1} \sim t$, $\dot{x}_{q1} \sim t$, $x_{q2} \sim t$, $\dot{x}_{q2} \sim t$, $u \sim t$, $H_d \sim t$)

Some points have been discussed in subsection (5.5.3) regarding tuning the parameters for applying the controller on practical system. The following observations are made from the simulation results shown in Figures 5.1 and 5.2..

1. Pendulum goes from vertical downward position to vertical upward position in one

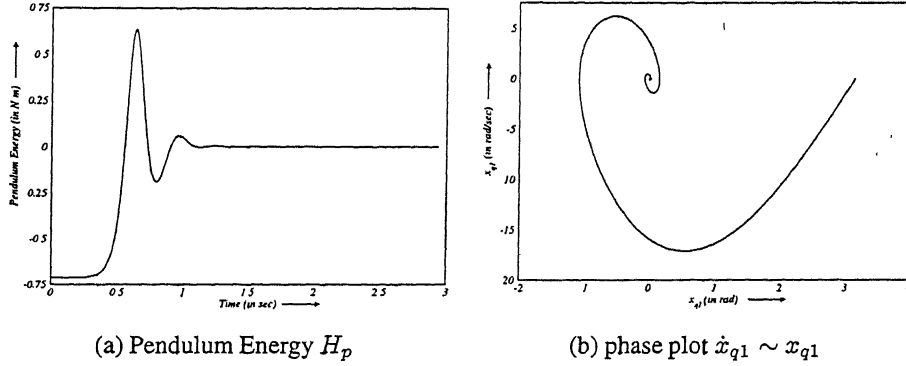


Figure 5.2: Simulation results of IDA-PBC ($H_p \sim t$, $\dot{x}_{q1} \sim x_{q1}$)

swing and stabilizes at the top with a damped oscillation.

2. Pendulum angular velocity finally converges to zero.
3. From the plot of wheel angle for a period of 3 seconds, it appears to attain a final value of 1600 radian, but actually it converges to zero very slowly and may take a huge amount of time (> 5000 seconds).
4. Wheel angular velocity finally converges to zero, but the peak angular velocity becomes as large as 6000 radian/second and it requires a very large control input of about 175 unit. These value are difficult to achive in real time systems.
5. Closed loop energy is non increasing, which converges to zero as the pendulum reaches the vertical upright position.

Limitations

The simulation results for IDA-PBC to bring the pendulum from vertical downward position to vertical upward position, are shown in figures 5.1-5.2. The figure 5.1(e) shows that the maximum value of control input required in this case is ± 110 , which is beyond the specified limit of ± 10 , hence global stabilization can not be achieved on practical system by this controller *i.e.* the downward equilibrium point $[\pi, \pi, 0, 0]$ does not lie in the domain of attraction of vetrical upward equilibrium point with limit on the control input.

Then what is the domain of attraction of upright equilibrium point with limit on control input ?. From the simulations and experiments it is determined that with the initial states $[x_{q1} = \pm 0.12 \text{ rad}, x_{q2} = \mp 1600 \text{ rad}, \dot{x}_{q1} = \mp 1.6 \text{ rad/sec}, \dot{x}_{q2} = \pm 300 \text{ rad/sec}]$, vertical upward position is asymptotically stable with the limit on the control input. Hence this controller can be used as a balancing controller. If the initial conditions are within the above specified range then the pendulum finally goes to the upright position. The simulation results for balancing control are shown in figures [5.3-5.5] along with experimental results. The figures [5.4(e), 5.4(f)] clearly shows that the control input ($|u| < 10$) is within specified limit, which brings the pendulum from $x_{q1} = 0.12 \text{ rad}$ to vertical upward position $x_{q1} = 0.0 \text{ rad}$.

5.5.4 Simulation and Experimental results of Balancing IWP by IDA-PBC

Due to the above said limitations on practical system, here IDA-PBC is used only for balancing control. For swing up a modified controller is used, which is discussed in next subsection with simulation and experimental results. Both simulation and Experimental results for balancing the pendulum are obtained with the same parameters tuned in subsection (5.5.3). While tuning the parameters in addition to satisfying conditions (5.46) and (5.53) following points should be kept in mind, (i) With constant a and k_v , as d decreases (*i.e. increases in negative direction*) settling time as well as overshoots of pendulum angle x_{q1} decreases. (ii) With constant d and k_v as a decreases, time interval between consecutive zero crossing of pendulum angle x_{q1} decreases, which is required to damp out the small oscillation around top position during stabilisation. (iii) with constant a and d if k_v increases, damping injection increases, which ultimately decreases settling time. From the practical observation it is observed that during stabilisation, time interval between two consecutive zero crossing of pendulum angle x_{q1} should be $< 0.35 \text{ sec}$ for better performance of stabilisation. The simulation and experimental results are compared in figures 5.3, 5.4 and 5.5. Here are some close observations of the results.

1. The pendulum angle exactly converges to zero in simulation, but there is a small error in experimental result due to offset in angle measurement by optical encoder.

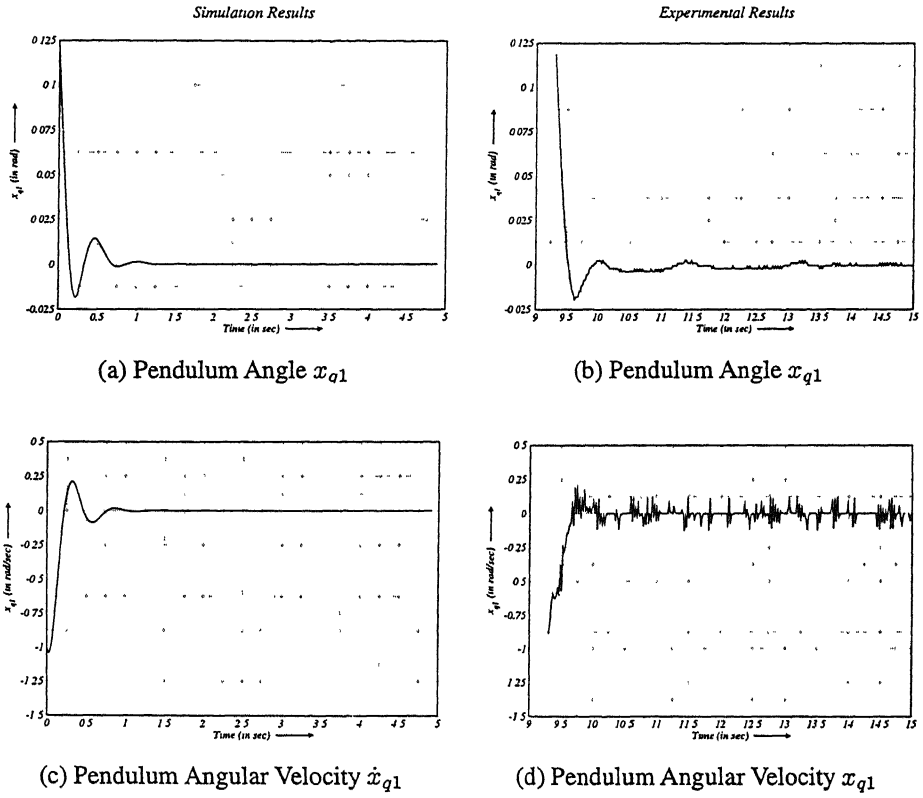


Figure 5.3: Simulation and Experimental Results of Balancing IWP by IDA-PBC ($x_{q1} \sim t$, $\dot{x}_{q1} \sim t$)

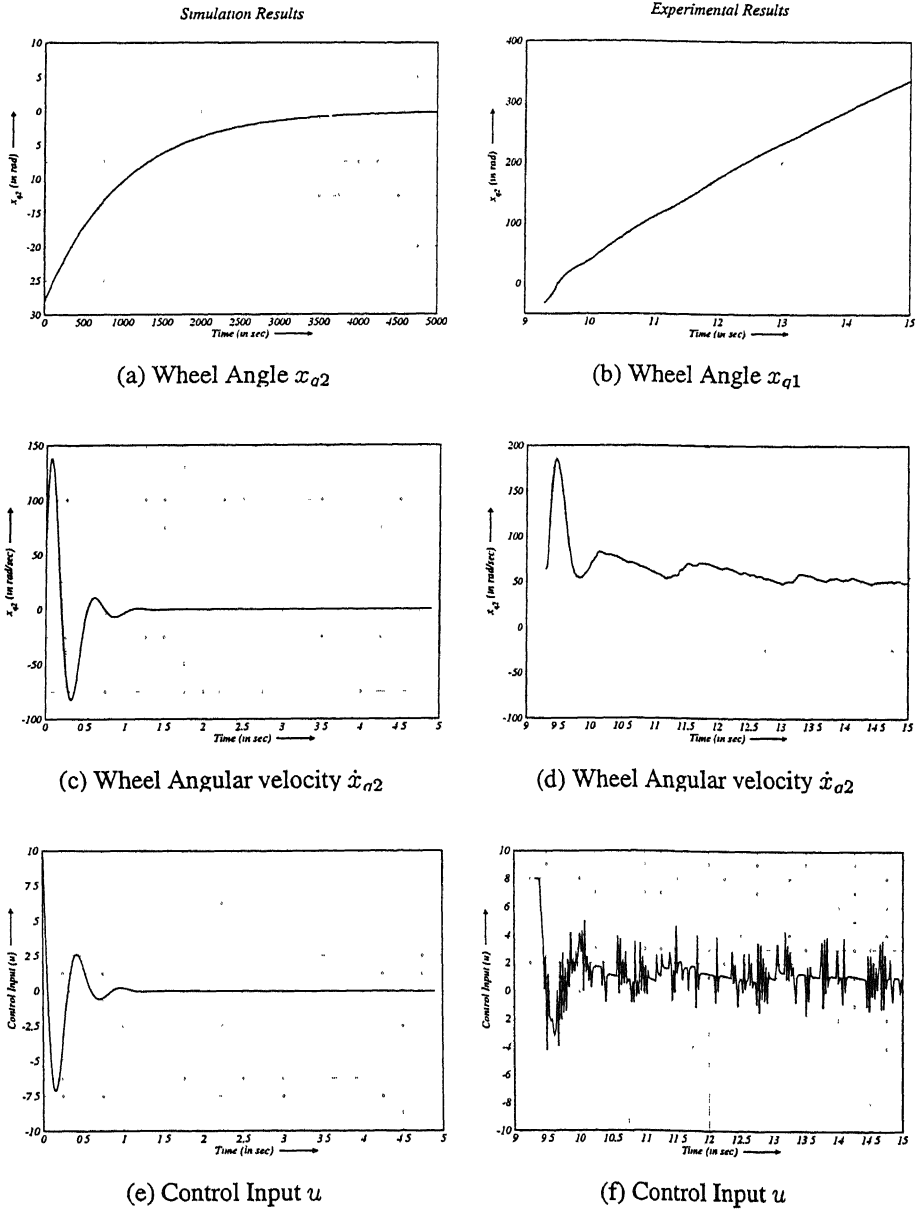


Figure 5.4: Simulation and Experimental Results of balancing IWP by IDA-PBC ($x_{q2} \sim t$, $\dot{x}_{q2} \sim t$, $u \sim t$)

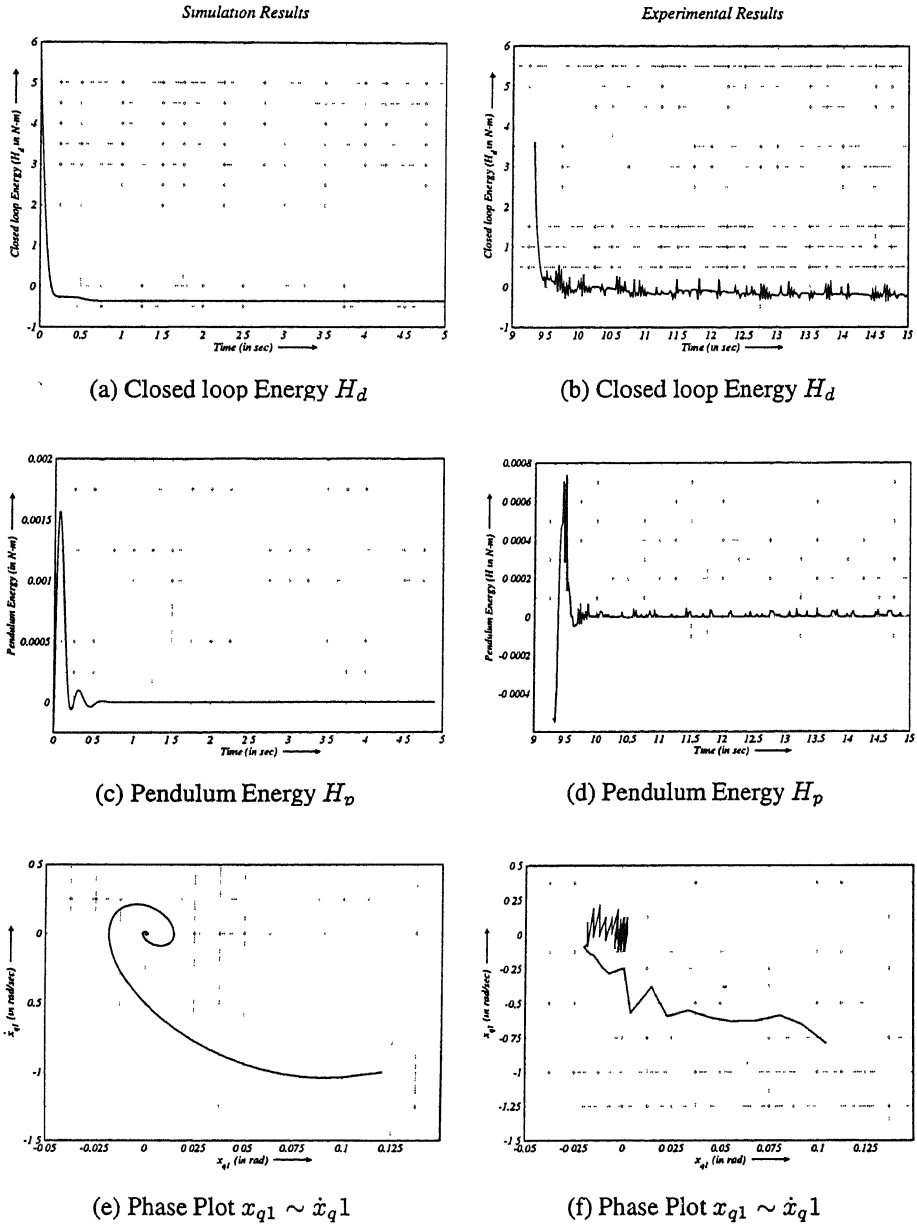


Figure 5.5: Responses of balancing IWP by IDA-PBC ($H_d \sim t$, $H_p \sim t$, $\dot{x}_{q1} \sim x_{q1}$)

2. The pendulum angular velocity converges to zero in simulation, but in experiment, it fluctuates around zero due to amplification of measurement noise in pendulum angle.
3. The wheel angle converges to zero in simulation after a large time of 5000 seconds, but in experiment, it is not possible to collect data for such a long time due to hardware limitations. We could collect data for only 4 seconds. But still, one can observe that the wheel angle is slowly increasing.
4. The Wheel speed should converge to zero as per the simulation results, but in real time experiment it settles to some steady state value.
5. The control input is very smooth in simulation but experimental results show a lot of noise.
6. Closed loop energy plot of both simulation and experimental results are quite similar. The energy decreases and attains a steady state value as soon as pendulum reaches the vertical upright position.

5.5.5 Swing up Controller

to bring the pendulum from vertical downward position to vertical upward position and stabilize it at upright position. From last subsection it is clear that due to limit on control input, with IDA-PB controller a small region of pendulum angle ($x_{q1} \leq \pm 0.12$) lie in the domain of attraction of upward equilibrium point. So the controller cannot bring the pendulum from downward position to upward position. To do the job it is required to design a swingup controller, which can bring the pendulum from vertical downward position to the domain of attraction of the vertical upward position. Once the pendulum reaches into the domain of attraction of vertical upward position, switching can be done from swingup to balancing controller, which stabilizes the pendulum asymptotically about upward position. As per the design of IDA-PBC a damping is injected to the closed loop energy, by which the closed loop energy H_d converges to zero, at stable equilibrium point. With no limit on control input, as the pendulum moves from downward position towards upward position, the closed loop energy H_d decreases with the system energy H increases from $-2mgl$ at downward position to 0 at top position. In

this subsection to design swing up controller we concentrate on the pendulum energy H instead of closed loop energy H_d . The control law derived in equation (5.59) is rewritten here for convenience,

$$u = \underbrace{\gamma_1 \sin x_{q1} + K_p(x_{q1} + \gamma_2 x_{q2})}_{\text{Energy Shaping}} + \underbrace{K_v \left(x_{p1} + \gamma_2 \frac{j}{j_r} x_{p2} \right)}_{\text{Damping Injection}}$$

The first term shapes both kinetic and potential energy of closed loop system and second term injects damping. Second term is a negative feedback of closed loop system output which damps out the closed loop system energy, such that the closed loop energy converges from a positive value to 0. Now lets see the effect of positive output feedback. With positive feedback the control input becomes,

$$u = \gamma_1 \sin x_{q1} + K_p(x_{q1} + \gamma_2 x_{q2}) - K_v \left(x_{p1} + \gamma_2 \frac{j}{j_r} x_{p2} \right)$$

with this time derivative of closed loop energy $\dot{H}_d$ comes out to be,

$$\dot{H}_d = \frac{K_p' K_v}{j K_p} \left(x_{p1} + \gamma_2 \frac{j}{j_r} x_{p2} \right)^2 \quad (5.67)$$

Equations (5.60) and (5.67) imply that H_d is a non-decreasing function. This is the case for unconstrained input case. With limit on the control input the above equation becomes too complex, from which we can not infer about the sign definiteness of the function H_d . As we intend to design swing up control, which requires shaping of pendulum energy $H_p = \frac{1}{2} \frac{x_p^2}{j} + mgl(\cos x_{q1} - 1)$ System energy H excluding rotor energy, we will consider only pendulum energy H_p . So we do not require the closed loop energy shaping term. Neglecting this term from control input, we get

$$u = -K_v \left(x_{p1} + \gamma_2 \frac{j}{j_r} x_{p2} \right) \quad (5.68)$$

which is the positive output feedback term. With this rate of pendulum energy,

$$\dot{H}_p = K_v \frac{x_{p1}}{j} \left(x_{p1} + \gamma_2 \frac{j}{j_r} x_{p2} \right) \quad (5.69)$$

With limit on input the above equation becomes

$$\dot{H}_p = -\frac{x_{p1}}{j} \times \text{sat} \left[-K_v(x_{p1} + \gamma_2 \frac{j}{j_r} x_{p2}) \right] \quad (5.70)$$

where $\text{sat}[u]$ is defined as,

$$\text{sat}[u] = \begin{cases} U & \text{if } u > U \\ -U & \text{if } u < -U \\ u & \text{otherwise} \end{cases} \quad (5.71)$$

where U is a positive quantity, $+U$ and $-U$ are the maximum and minimum limit on control input, respectively. Due to presence of term $\gamma_2 \frac{j}{j_r} x_{p2}$ In equation (5.70), we can not say anything about the sign definiteness of function $\dot{H}_p$. So we modify the control law as follows

$$u = -K_v(x_{p1}) \quad (5.72)$$

one can see that this is no longer a positive output feedback control law, rather a positive partial state feedback control law, in which proportional of only one state *i.e.* x_{p1} is feedback. With this modified control law rate of pendulum energy takes the form,

$$\begin{aligned} \dot{H}_p &= K_v \frac{x_{p1}^2}{j} && \text{with no limit on control input} \\ \dot{H}_p &= -\frac{x_{p1}}{j} \times \text{sat}[-K_v(x_{p1})] && \text{with limit on control input} \end{aligned} \quad (5.73)$$

So equation (5.72) implies that H_p is a non-decreasing function. Hence the positive partial state feedback control injects energy into the system. But it does not tell us that the state x_{p1} will remain bounded. To keep x_{p1} bounded, the pendulum energy must be kept bounded. So we put a bound $H_p \leq 0$, where $H_p = 0$ at top position of the pendulum. This bound on H_p is put with the control law

$$u = \begin{cases} -K_v(x_{p1}) & \text{if } H_p \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad (5.74)$$

With this control law Pendulum energy H_p reaches to 0, it then follows that

$$H_p = \frac{1}{2} \frac{x_{p1}^2}{j} + mgl(\cos x_{q1} - 1) = 0$$

Since j is a constant, the above equation defines the homoclinic orbit of the pendulum in $(x_{q1}, \dot{x}_{q1})$ phase plane. Hence the pendulum moves clockwise or anti-clockwise until it reaches to top position.

5.5.6 Simulation and Experimental results for Complete stabilisation

Simulation and experimental results for complete stabilisation with combined control are shown in figures 5.6-5.8. The control parameters, tuned in subsection (5.5.3) are used for the controller design. Figures 5.6(a), 5.6(b) shows the plot of pendulum angle x_{q1} with time. With swing up algorithm the pendulum oscillates with increasing amplitude until x_{q1} reaches within the range $\pm 0.12 \text{ rad}$. Due to bound on energy and limit on input, state variables $\dot{x}_{q1}$ and $\dot{x}_{q2}$ reaches into the domain of attraction of. The energy H_p of the pendulum increases from $-0.7133 \text{ N} - m$ to $0 \text{ N} - m$ as it approaches upward position from downward position. The closed loop energy H_d , shown in figure [5.8(a), 5.8(b)] varies in a different manner. The envelop of the closed loop energy increases, the lower boundry of this is established by closed loop energy at the instants pendulum passes through vertical downward position and upper boundary by the closed loop energy at the instants pendulum goes to peak in clockwise or anti-clockwise direction in each swing. It reaches to 0 when pendulum reaches top position. One thing is clear from the closed loop energy plot, if we consider one swing from downward position upto a peak in any direction, the closed loop energy is increasing, which is nothing but the effect of positive state feed back control input. As soon as the pendulum reaches within the domain of attraction, we switch over to balancing control which stabilises the pendulum about top position.

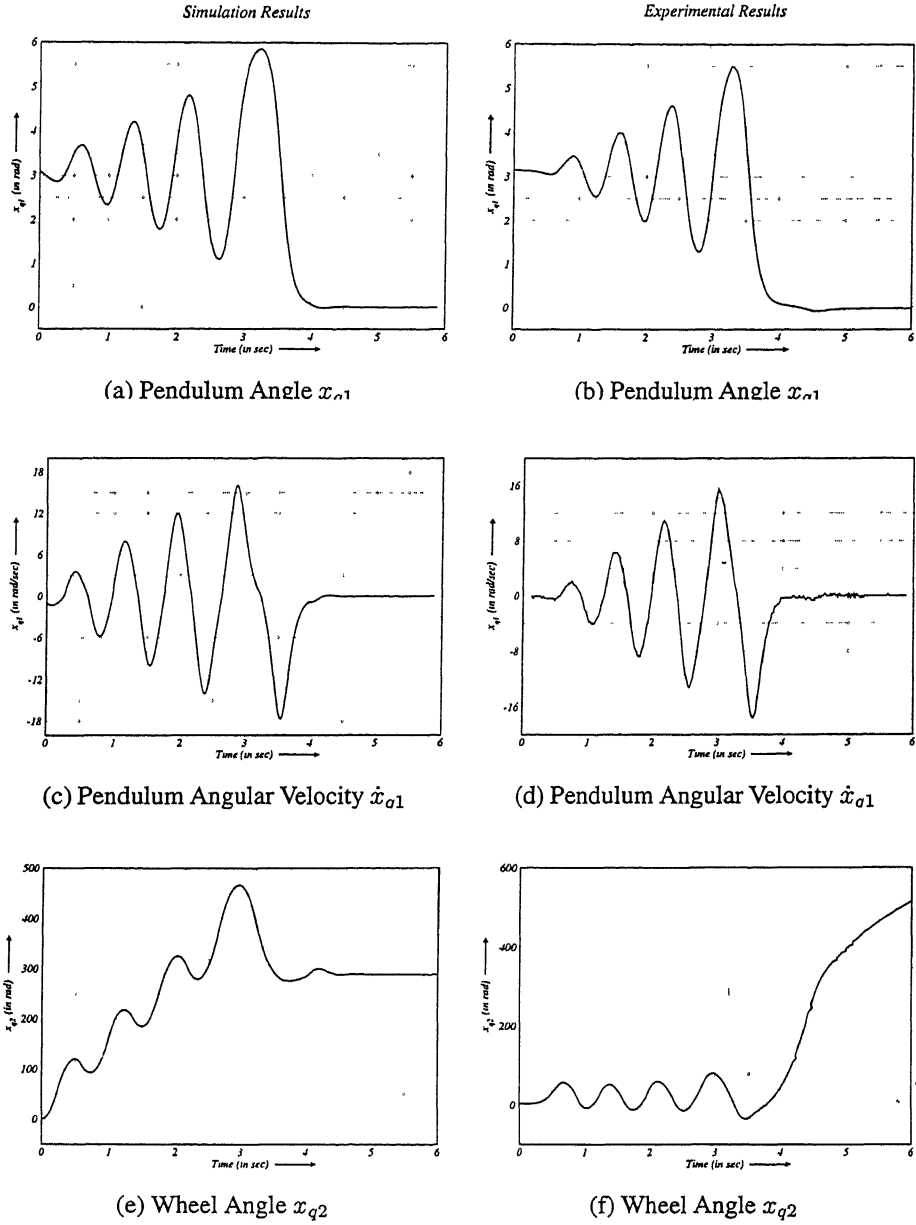


Figure 5.6: Responses of Complete Stabilization ($x_{q1} \sim t$, $\dot{x}_{q1} \sim t$, $x_{q2} \sim t$)

5.5.7 Concluding Remarks

The IDA-PBC is successfully implemented on inertia wheel pendulum for stabilising it about vertical upward position. The real complexity of the IDA-PBC design methodology lies in solvability of PDEs. The additional matrix $R_a(x)$ for IWP is chosen to make the solution of

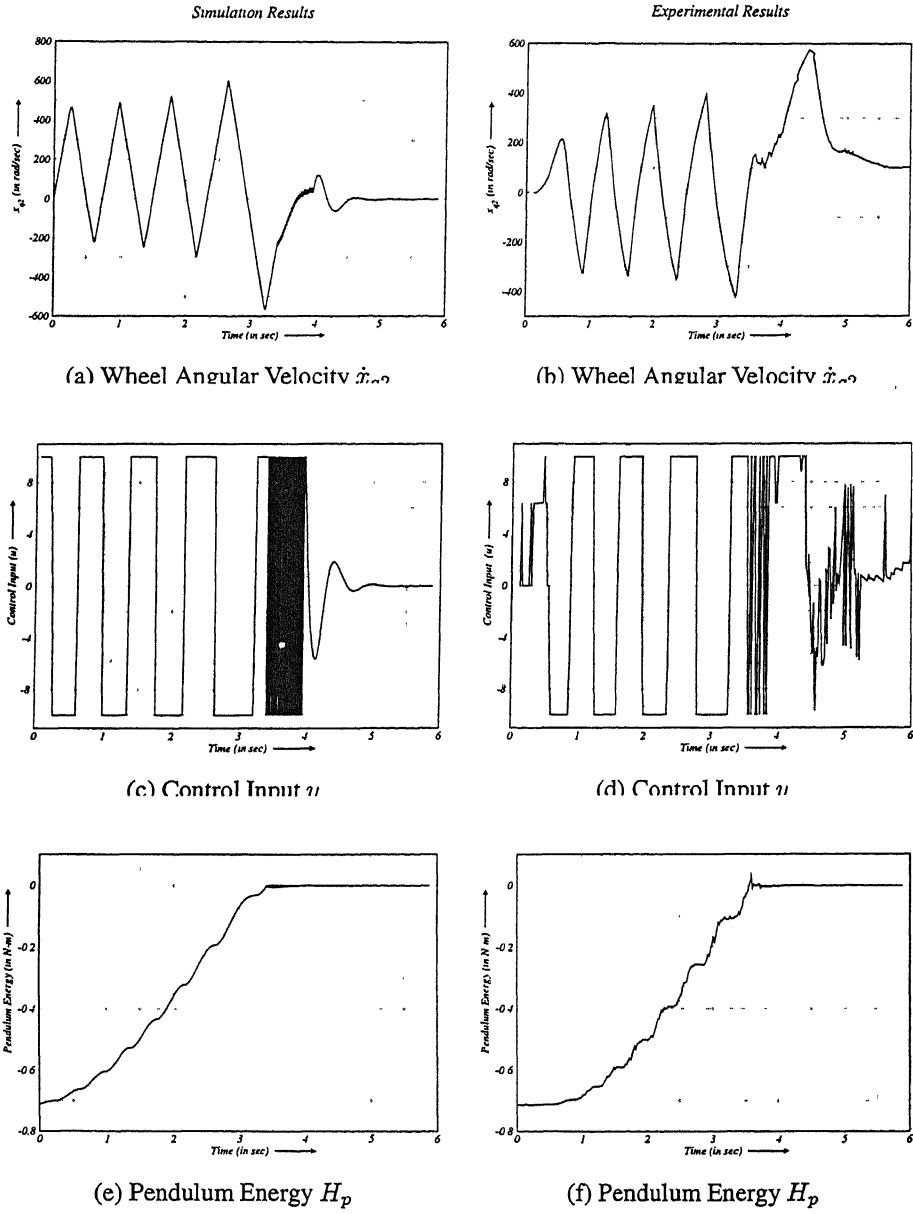


Figure 5.7: Responses of Complete Stabilisation ($\dot{x}_{q2} \sim t$, $u \sim t$, $H_p \sim t$)

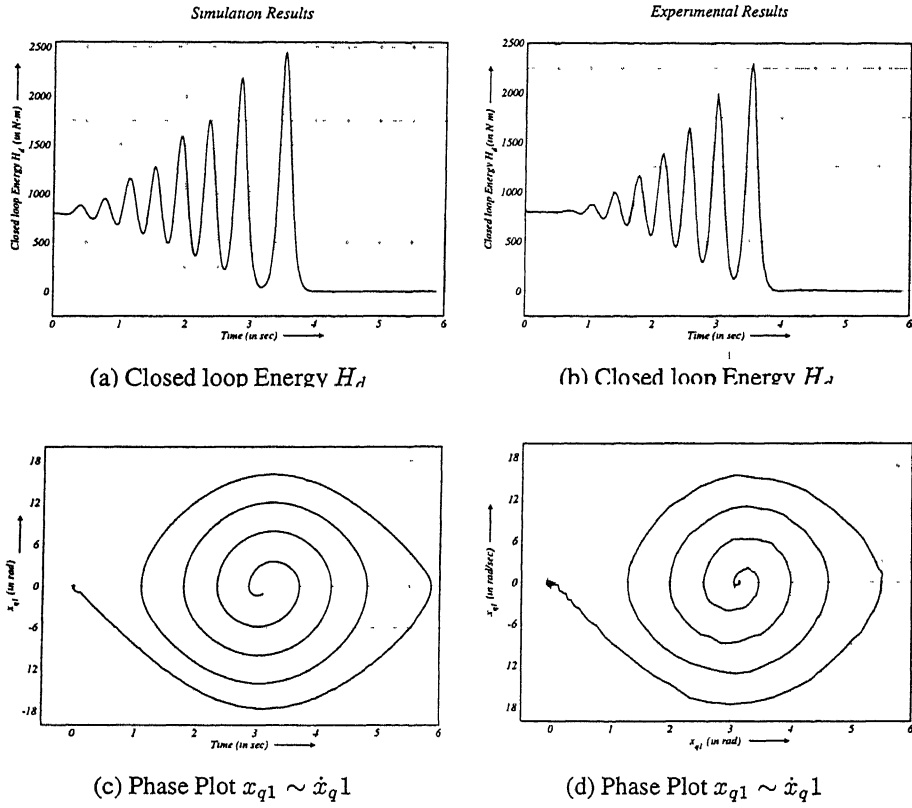


Figure 5.8: Responses of Complete Stabilization ($H_d \sim t$, $\dot{x}_{q1} \sim x_{q1}$)

PDE simple. The matrix $R_a(x)$ came directly from negative output feedback damping term of control input u . With IDA-PBC method a balancing controller was designed to stabilise the pendulum at top position. The Hamiltonian structure is preserved in closed loop by this control. For the limited input practical system case the stable equilibrium point is not globally stabilizable with IDA-PBC method, so a novel swing up controller is designed with some modification in the balancing control, which is a positive state feedback control instead of output feedback. In this control, some property of Hamiltonian structure is preserved which was observed in closed loop energy function for a portion of pendulum swing. One advantage of this combined algorithm is that, it is much faster than the other two control techniques implemented in previous chapters.

* * *

Chapter 6

Conclusion

In this chapter we make concluding remarks and state the scope for future work.

6.1 Summary

In this thesis, various kinds of dual mode controllers for inertial wheel pendulum (IWP) have been studied, analyzed and established through real time experimentation. The emphasis has been on energy based controllers. Some of the contributions of this work are as follows:

- A dual mode control scheme based on feedback linearization is revisited. Here, the swing up is achieved by partial feedback linearisation control and stabilisation is achieved by input-state feedback linearization.
- A modification has been done to the existing passivity based swing up control which enables the pendulum to reach the domain of attraction of balancing controller faster. Real time experimental results show a marked improvement in the performance of the modified algorithm over the partial feedback linearisation based swing up control.
- A novel energy based swing up controller in conjunction with IDA-PBC stabilizing controller has been designed for the IWP. During experimentation with Mechatronics kit, this controller was found to be the fastest among all three dual mode controllers. The stability and convergence of this algorithm has been established through extensive derivations. The experimental results corroborate our theoretical findings.

6.2 Scope of Future Work

- All the controllers, that have been implemented here, are dual mode controllers. So one may think of implementing a single algorithm that can achieve both swing up and balancing. The IDA-PBC is a single controller that achieves the above objective only in simulation. It does not work on real time systems as it requires very high control input and disk speed. One can investigate deeper into this algorithm to make it work on the real time systems.
- Motor dynamics has not been taken into account in any of the above mentioned algorithms. By considering motor dynamics, robust controllers can be designed which are insensitive to parameter variations and external disturbances.
- In our work, the system parameters were available to us, which may not be so in many cases. Adaptive controllers may be designed to estimate these parameters so that system independent controllers can be designed.

* * *

Bibliography

- [1] Mark W. Spong, Peter Corke, and Rogelio Lozano. Nonlinear control of the reaction wheel pendulum. *Automatica*, 37:1845–1851, 2001.
- [2] M. Vidyasagar. *Nonlinear Systems Analysis*. Printice Hall, 1992.
- [3] A. Isidori. *Nonlinear Control systems: An Introduction*. Springer-Verlag Berlin, 3rd. edition, 1995.
- [4] Jean-Jacques E. Slotine and Weiping Li. *Applied Nonlinear Control*. Prentice Hall, 1991.
- [5] V. I. Utkin. Variable structure systems with sliding modes. *IEEE Transactions on Automatic Control*, AC-22:212–222, 1977.
- [6] K. D. Young and S. V. Drakunov. Discontinuous frequency shaping compensation for uncertain dynamic systems. In *Proc. 12th IFAC World Congr.*, pages 39–42, Sydney, Australia, 1993.
- [7] M. Kristic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear And Adaptive Control Design*. John Wiley & Sons, Inc, 1995.
- [8] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse. Systematic design of adaptive controllers for feedback linearizable systems. *IEEE Transactions on Automatic Control*, 36:1241–53, 1991.
- [9] Chiman Kwan and F. L. Lewis. Robust backstepping control of nonlinear systems using neural networks. *IEEE Transactions on Systems, Man, and Cybernetics*, 30(6):753–756, 2000.

- [10] Antonio Loria and Henk Nijmeijer. Passivity based control.
- [11] Willems J.C. Dissipative dynamical systems part ii: linear systems with quadratic supply rate. In *Archieve for Rotational Mechanics and Analysis*, volume 45, pages 352–393, 1972.
- [12] Wu M. and Desoer C.A. Input-output propoities of multiple-input, multiple-output discrete systems: part ii. In *Journal of the Franklin Institute*, volume 290, pages 85–101, 1970.
- [13] Vidyasagar M. L2 stability of interconnected systems using a reformulation of the passivity theorem. *IEEE Tran. Circuits and Systems*, CAS-24(11):637–645, 1977.
- [14] Zames G. On the input-output stability of time varying nonlinear feedback systems. parti: conditions derived using concepts of loop gain, conicity and positivity. *IEEE Tran. on Auto. Control*, AC-11(2):465–476, 1966.
- [15] Hill D.J. and Moylan P. Stability results for nonlinear feedback systems. *Automatica*, 13:377–382, 1977.
- [16] Byrnes C.I., Isidori A., and Willems J.C. Passivity feedback equivalence and the global stabilisation of minimum phase nonlinear systems. *IEEE Trans. on Auto. Control*, 36:1228–1240, 1991.
- [17] Kokotovic P.V. and Sussmann H.J. A positive real condition for global stabilisation of nonlinear systems. In *Systems and Control Letters*, volume 13, pages 125–133, 1989.
- [18] Selpulchre R. and Jankovic M. and Kokotovic P. *Constructive Non-linear Control*. Springer-Verlag, London, 1997.
- [19] Santosusso G.L. Passivity of nonlinear systems with input-output feedthrough. *Automatica*, 33(4):693–697, 1997.
- [20] Picci G. and Pinzoni S. Feedback dissipative systems. In *Journal of Math Systems and control*, volume 2, pages 1–30, 1992.

- [21] Lin W. and Byrnes. Passivity and absolute stabilization of a class of discrete time non-linear systems. *Automatica*, 31(2):263–267, 1995.
- [22] Ortega R., Loria A., Nicklasson P.J., and Sira Ramrez. *Passivity control of EL systems, Mechanical, Electrical and Electromechanical Applications*. Springer, 1998.
- [23] Khalil K. *Nonlinear Systems*. Prentice Hill, 1996.
- [24] Isabelle Fantoni and Rogelio Lozano. *Non-linear Control for Underactuated Mechanical Systems*. Springer, 2002.
- [25] Frederic Mazenc and Laurent Praly. Adding integrations, saturated controls, and stabilization for feedforward systems. *IEEE Tran. on Automatic Control*, 41:1559–1577, 1996.
- [26] M. W. Spong. Modelling and control of elastic joint robots. *ACMS journal of Dynamic systems, Measurement and Control*, 109:310–319, 1987.
- [27] N. E. Leonard. Periodic forcing, dynamics and control of underactuated spacecraft and underwater vehicles. In *Proc. 34th Conf. on decision and control*, pages 3980–3985, New Orleans, LA, 1995. IEEE.
- [28] Reza Olfati Saber. Global stabilization of a flat underactuated system: the inertia wheel pendulum. In *IEEE Conference on Decision and Control*, Orlando, FL, 2001.
- [29] Romeo Ortega and Mark W. Spong. Stabilisation of underactuated mechanical systems via interconnection and damping assignment. *IFAC Workshop of Lagrangian and Hamiltonian methods in Nonlinear Systems*, March 2000.
- [30] K. Astrom, D. Block, and Mark W. Spong. *The Reaction Wheel Pendulum*. Quanser Mechatronics kit.
- [31] Mark W. Spong. The swing up control problem for the acrobot. *IEEE Control Systems Magazine*, 15:49–55, Feb 1995.

- [32] Mark W. Spong. Partial feedback linearization of underactuated mechanical systems. *IROS '94*, pages 314–321, Sep 1994.
- [33] R.Ortega and M.W.spong. Adaptive motion control of rigid robots : A tutorial. *Automatica*, 25(6):877–888, 1989.
- [34] Tomas Nilson and Miguel Perez. Introduction to passivity-based control of euler-lagrange systems. Technical report, Vaxjo University, November 2003.
- [35] Romeo Ortega, Arjan van der Schaft, Bernhard Maschke, and Gerardo Escobar. Interconnection and damping assignment passivity-based control of port-controlled hamiltonian systems. *Automatica*, 38:585–596, 2002.
- [36] Romeo Ortega, Mark W. Spong, Fabio Gomez-estern, and Guido Blankenstein. Stabilisation of a class of underactuated mechanical system via interconnection and damping assignment. *IEEE Transaction on Automatic Control*, 47, NO. 8:1218–1233, August 2002.
- [37] Ajan Van der Schaft. Port-controlled hamiltonian systems:towards theory for control and design of nonlinear physical systems. *SICE, Japan*, 39, no.2:91–98, November 2000.
- [38] Guido Blankenestein, Romeo Ortega, and Ajan Van der Schaft. The matching conditions of controlled lagrangian and intrconnection and damping assignment passivity based control. *Int. Journal Control*, 2001.